Chapter 4

The KdV problem in domains that can be transformed into rectangles

In this chapter, we study the Korteweg-de Vries equation subject to boundary condition in non-rectangular domain, with some assumptions on the boundary of the domain and the coefficients of equation. The right-hand side and its derivative with respect to t are in the Lebesgue space L^2. The goal is to establish the existence, the uniqueness and the regularity of the solution.

4.1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be the domain

$$\Omega = \{(t,x) \in \mathbb{R}^2 : 0 < t < T,\ x \in I_t\},$$
$$I_t = \{x \in \mathbb{R} : \varphi_1(t) < x < \varphi_2(t),\ t \in (0,T)\},$$

we consider the semi-linear Korteweg-de Vries problem with time variables coefficients

$$\begin{cases} \partial_t v(t,y) + a(t)v(t,y)\partial_y v(t,y) + b(t)\partial_y^3 v(t,y) = g(t,y) \ \text{ in } \Omega, \\ v(0,y) = 0, \quad y \in I_0, \\ v(t,\varphi_1(t)) = v(t,\varphi_2(t)) = \partial_y v(t,\varphi_2(t)) = 0 \quad t \in (0,T), \end{cases} \tag{4.1}$$

where $I_0 =]\varphi_1(0), \varphi_2(0)[$ and $a(t)$, $b(t)$ are given. g and $\partial_t g$ are in $L^2(\Omega)$.

We are particularly interested in the question: What conditions on the functions $a(t)$, $b(t)$ and $(\varphi_i(t))_{i=1,2}$ guarantee that (4.1) admits a unique solution u.

Consider the case where $a(t)$, $b(t)$, $a'(t)$ and $b'(t)$ are positive and bounded functions for all $t \in [0,T]$. Let h be defined by

$$\begin{array}{rcl} h : [0,T] & \to & [0,T'] \\ t & \mapsto & h(t) = \displaystyle\int_0^t b(s)\mathrm{d}s, \end{array}$$

we put $\psi_i = \varphi_i \circ h^{-1}$ where $i = 1,2$. Using the change of variables $t' = h(t)$, $w(t',y) = v(t,y)$, (4.1) can be written in the form

$$\begin{cases} \partial_{t'} w(t',y) + d(t')w(t',y)\partial_y w(t',y) + \partial_y^3 w(t',y) = h(t',y) \quad (t',y) \in \Omega', \\ w(0,y) = 0, \\ w(t',\psi_1(t')) = w(t',\psi_2(t')) = \partial_y w(t',\psi_2(t')) = 0 \quad t' \in (0,T'), \end{cases}$$

where $d(t') = \dfrac{a(t)}{b(t)}$, $h(t',y) = \dfrac{g(t,y)}{b(t)}$, $\Omega' = \{(t',x) \in \mathbb{R}^2;\ 0 < t' < T',\ x \in I_{t'}\}$ and $T' = \displaystyle\int_0^T b(s)\mathrm{d}s$.

So, to solve (4.1), it is enough to solve the following problem

$$\begin{cases} \partial_t v(t,y) + c(t)v(t,y)\partial_y v(t,y) + \partial_y^3 v(t,y) = g(t,y) \quad (t,y) \in \Omega, \\ v(0,y) = 0 \quad y \in I_0, \\ v(t,\varphi_1(t)) = v(t,\varphi_2(t)) = \partial_y v(t,\varphi_2(t)) = 0 \quad t \in (0,T). \end{cases} \tag{4.2}$$

In the sequel, we assume that there exist positive constants c_1 and c_2, such that

$$\begin{aligned} c_1 \le c(t) \le c_2, \quad \text{for all } t \in [0,T]. \\ c_1 \le c'(t) \le c_2, \quad \text{for all } t \in [0,T]. \end{aligned} \tag{4.3}$$

We also impose the hypothesis

$$\varphi_1',\ \varphi_2',\ \varphi_1'' ,\varphi_2'' ,\varphi_1''' ,\varphi_2''' \in L^1(0,T) \tag{4.4}$$

to establish the existence and uniqueness of a solution to (4.1).

Then, for all $t \in [0,T]$, we have

$$|\varphi_1(t)| \le |\varphi_1(0)| + \int_0^t |\varphi_1'(s)|\,\mathrm{d}s \le |\varphi_1(0)| + \int_0^T |\varphi_1'(s)|\,\mathrm{d}s \le \gamma,$$

where γ is a positive constant. Similarly, we have

$$\begin{aligned} &|\varphi_2(t)| \le \gamma, \quad |\varphi_1'(t)| \le \gamma, \quad |\varphi_2'(t)| \le \gamma, \\ &|\varphi_1''(t)| \le \gamma, \quad |\varphi_2''(t)| \le \gamma, \end{aligned} \quad \text{for all } t \in [0,T] \tag{4.5}$$

therefore

$$|\varphi(t)| \le 2\gamma, \quad |\varphi'(t)| \le 2\gamma, \quad |\varphi''(t)| \le 2\gamma \quad \text{for all } t \in [0,T], \tag{4.6}$$

where $\varphi(t) = \varphi_2(t) - \varphi_1(t)$.

The change of variables:

$$\begin{aligned} \Omega &\to R \\ (t,y) &\mapsto (t,x) = \left(t, \frac{y-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}\right), \end{aligned} \tag{4.7}$$

transforms Ω into the rectangle $R = (0,T) \times I$ where $I = (0,1)$. Putting $v(t,y) = u(t,x)$ and $g(t,y) = f(t,x)$, Problem (4.2) becomes

$$\begin{cases} \partial_t u(t,x) + \dfrac{c(t)}{\varphi(t)} u(t,x)\partial_x u(t,x) + \dfrac{1}{\varphi^3(t)}\partial_x^3 u(t,x) - \dfrac{x\varphi'(t)+\varphi_1'(t)}{\varphi(t)}\partial_x u(t,x) \\ = f(t,x) \quad (t,x) \in R, \\ u(0,x) = 0 \quad x \in I, \\ u(t,0) = u(t,1) = \partial_x u(t,1) = 0 \quad t \in (0,T), \end{cases}$$

multiplying both sides of the equation of the last problem by $\varphi^3(t) > 0$, we obtain

$$\begin{cases} p(t)\partial_t u(t,x) + q(t)u(t,x)\partial_x u(t,x) + \partial_x^3 u(t,x) + r(t,x)\partial_x u(t,x) \\ = f(t,x) \quad (t,x) \in R, \\ u(0,x) = 0 \quad x \in I, \\ u(t,0) = u(t,1) = \partial_x u(t,1) = 0 \quad t \in (0,T), \end{cases} \tag{4.8}$$

where

$$p(t) = \varphi^3(t), \quad q(t) = \varphi^2(t)c(t) \quad \text{and} \quad r(t,x) = -\varphi^2(t)(x\varphi'(t) + \varphi_1'(t)).$$

It is easy to see that by change of variables (4.7) we have

$$\begin{aligned} & g \in L^2(\Omega) \Leftrightarrow f \in L^2(R) \\ & v \in L^\infty(0,T,L^2(I_0)) \Leftrightarrow u \in L^\infty(0,T,L^2(I)) \\ & v \in L^2(0,T,H^3(I_0)) \Leftrightarrow u \in L^2(0,T,H^3(I)) \end{aligned} \tag{4.9}$$

Remark 4.1. *According to* (4.3), (4.5) *and* (4.6), *the functions p, q and r satisfy the following assumptions*

$$\begin{cases} \alpha < p(t) < \beta, \quad \alpha < p'(t) < \beta \quad \forall t \in [0,T], \\ \alpha < q(t) < \beta, \quad \alpha < q'(t) < \beta \quad \forall t \in [0,T], \\ |r(t,x)| \le \beta, \ |\partial_x r(t,x)| \le \beta, \ \text{and } |\partial_t r(t,x)| \le \beta \quad \forall (t,x) \in R \end{cases} \tag{4.10}$$

where α and β are positive constants.

Our main result is :

Theorem 4.2. *If f, $\partial_t f$ are in $L^2(R)$, and p, q, r satisfy assumption* (4.10)*, Problem* (4.8) *admits a unique solution such that*

$$u \in L^\infty(0,T,L^2(I)) \cap L^2(0,T,H^3(I)),$$
$$\partial_t u \in L^\infty(0,T,L^2(I)) \cap L^2(0,T,H^1(I)).$$

Corollary 4.3. *If g, $\partial_t g + \nu(t,y)\partial_y g$ are in $L^2(\Omega)$, and $c(t)$, $\varphi_1(t)$, $\varphi_2(t)$ satisfy assumptions* (4.3) *and* (4.4)*, Problem* (4.1) *admits a unique solution such that*

$$v \in L^\infty(0,T,L^2(I_0)) \cap L^2(0,T,H^3(I_0)),$$
$$\partial_t v \in L^\infty(0,T,L^2(I_0)) \cap L^2(0,T,H^1(I_0)).$$

where $\nu(t,y) = \dfrac{\varphi'(t)y - \varphi_1\varphi_2' + \varphi_1'\varphi_2}{\varphi(t)}$.

We will prove Theorem 4.2 , and thanks to the change of variables (4.7) and (4.9) the Corollary 4.3 will be a direct result of Theorem 4.2 .

4.2 Preliminaries

Lemma 4.4. *There exist eigenfunctions of the following problem*

$$\begin{cases} e_j^{(4)} = \lambda_j e_j, \quad j \in \mathbb{N}^*, \\ e_j(0) = e_j(1) = e_j''(0) = e_j'(1) = 0. \end{cases}$$

which create a basis in $H^4(I)$ orthonormal in $L^2(I)$

Proof. For every $f \in L^2(I)$ there exists a unique solution satisfying

$$\begin{cases} \partial_x^4 u = f, \\ u(0) = u(1) = u''(0) = u'(1) = 0. \end{cases} \tag{4.11}$$

Denote by T the operator $f \longmapsto u$ considered as an operator from $L^2(I)$ into $L^2(I)$.

We claim that T s self-adjoint and compact. First, the compactness. Because of (4.11) we have

$$\int_0^1 (\partial_x^2 u)^2 \,\mathrm{d}x = \int_0^1 f u \,\mathrm{d}x$$

and thus

$$\begin{aligned}\|\partial_x^2 u\|_{L^2(I)}^2 &\leq \|f\|_{L^2(I)}^2 \|u\|_{L^2(I)}^2 \\ &\leq C\|f\|_{L^2(I)}^2\end{aligned}$$

where C is a positive constant. This can be written as

$$\|Tf\|_{H^2(I)}^2 \leq C\|f\|_{L^2(I)}^2, \quad \forall f \in L^2(I).$$

Since the injection of $H^1(I)$ into $L^2(I)$ is compact (because I is bounded), we deduce that T is a compact operator from $L^2(I)$ into $L^2(I)$. Next, we show that T is self-adjoint, i.e.,

$$\int_0^1 (Tf)g \,\mathrm{d}x = \int_0^1 f(Tg)\,\mathrm{d}x, \quad \forall f, g \in L^2(I).$$

Indeed, setting $u = Tf$ and $v = Tg$, we have

$$-\partial_x^4 u = f \tag{4.12}$$

and

$$-\partial_x^4 v = g \tag{4.13}$$

Multiplying (4.12) by v and (4.13) by u and then integrating, we obtain

$$\int_0^1 \partial_x^2 u \partial_x^2 v \,\mathrm{d}x = \int_0^1 fv\,\mathrm{d}x = \int_0^1 gu\,\mathrm{d}x.$$

which is the desired conclusion.

Finally, we note that

$$\int_0^1 (Tf)f\,\mathrm{d}x = \int_0^1 uf\,\mathrm{d}x = \int_0^1 (\partial_x^2 u)^2\,\mathrm{d}x \geq 0, \quad \forall f \in L^2(I).$$

Hence, assertions of Lemma 4.4 follow from Theorem 6.1 [10]. □

4.3 Parabolic regularization

To prove Theorem 4.2 , we consider in $R = (0,T) \times (0,1)$ the approximated problem

$$\begin{cases} p(t)\partial_t u_\varepsilon(t,x) + q(t)u_\varepsilon(t,x)\partial_x u_\varepsilon(t,x) + \partial_x^3 u_\varepsilon(t,x) \\ \; + r(t,x)\partial_x u_\varepsilon(t,x) - \varepsilon\partial_x^2 u_\varepsilon + \varepsilon\partial_x^4 u_\varepsilon = f(t,x) \quad (t,x) \in R, \\ u_\varepsilon(0,x) = 0 \quad x \in (0,1), \\ u_\varepsilon(t,0) = u_\varepsilon(t,1) = \partial_x^2 u_\varepsilon(t,0) = \partial_x u_\varepsilon(t,1) = 0 \quad t \in (0,T). \end{cases} \tag{4.14}$$

In the case $r(t,x) = 0$ and $p(t)$, $q(t)$ are constants, the equation (4.14) becomes the KdV-Burgers-Kuramoto equation. For other choice of coefficients, (4.14) is a model for several problems that appear in different physical situations.

We establish the existence of a solution u_ε for (4.14), using the Faedo-Galerkin method. Then finding uniform estimates in ε we obtain Theorem 4.2 , as $\varepsilon \to 0$.

Theorem 4.5. *If f, $\partial_t f$ are in $L^2(R)$, and p, q, r satisfy assumption* (4.10)*, then* (4.14) *admits a solution such that*

$$u_\varepsilon \in L^\infty(0,T,H^2(I)) \cap L^2(0,T,H^4(I)),$$
$$\partial_t u_\varepsilon \in L^\infty(0,T,L^2(I)) \cap L^2(0,T,H^2(I)).$$

Proof. To prove the existence of a solution to (4.14), we choose the basis $(e_j)_{j\in\mathbb{N}^\star}$ of $H^4(I)$ (see Lemma 4.4), orthonormal in $L^2(I)$, defined as a subset of the eigenfunctions for the problem

$$\begin{cases} e_j^{(4)} = \lambda_j e_j, \quad j \in \mathbb{N}^*, \\ e_j(0) = e_j(1) = e_j''(0) = e_j'(1) = 0. \end{cases}$$

Then, we introduce the approximate solution $u_{\varepsilon n}$ by

$$u_{\varepsilon n}(t) = \sum_{j=1}^{n} c_{nj}(t)e_j, \tag{4.15}$$

which satisfies the approximate problem

$$\begin{cases} p(t)\int_0^1 \partial_t u_{\varepsilon n} e_j \,\mathrm{d}x + q(t)\int_0^1 u_{\varepsilon n}\partial_x u_{\varepsilon n} e_j \,\mathrm{d}x + \int_0^1 \partial_x^3 u_{\varepsilon n} e_j \,\mathrm{d}x \\ +\int_0^1 r(t,x)\partial_x u_{\varepsilon n} e_j \,\mathrm{d}x - \varepsilon\int_0^1 \partial_x^2 u_{\varepsilon n} e_j \,\mathrm{d}x + \varepsilon\int_0^1 \partial_x^4 u_{\varepsilon n} e_j \,\mathrm{d}x \\ = \int_0^1 f e_j \,\mathrm{d}x, \\ u_{\varepsilon n}(0) = u_{0n}. \end{cases} \tag{4.16}$$

for all $j = 1, ..., n$ and $0 \le t \le T$.

Problem (4.16) is equivalent to a system of n uncoupled nonlinear ordinary differential equations. So, this solution exists in some interval $(0, T')$ with $T' \le T$ (see for example [15]). The a priori estimate (4.17) that will be given, shows that $T' = T$. □

4.3.1 A priori estimates

Lemma 4.6. *There exists a positive constant K_1 independent of n and ε, such that for all $t \in [0, T]$*

$$\alpha\|u_{\varepsilon n}\|^2_{L^2(I)} + \varepsilon\int_0^t \|\partial_x^2 u_{\varepsilon n}(s)\|^2_{L^2(I)} \,\mathrm{d}s \le K_1, \tag{4.17}$$

where α is given in (4.10).

Proof. Multiplying (4.16) by c_j, taking the summation with respect to j from 1 to n, and using (4.15), we get

$$\frac{p(t)}{2}\frac{d}{dt}\int_0^1 u_{\varepsilon n}^2 \,\mathrm{d}x + \varepsilon\int_0^1 (\partial_x u_{\varepsilon n})^2 \,\mathrm{d}x + \varepsilon\int_0^1 (\partial_x^2 u_{\varepsilon n})^2 \,\mathrm{d}x$$

$$+\frac{1}{2}(\partial_x u(t,0))^2 - \frac{1}{2}\int_0^1 \partial_x r(t,x) u_{\varepsilon n}^2 \,\mathrm{d}x = \int_0^1 f u_{\varepsilon n} \,\mathrm{d}x.$$

Indeed, because of the boundary conditions, the integration by parts gives

$$q(t)\int_0^1 u_{\varepsilon n}^2\partial_x u_{\varepsilon n}\,\mathrm{d}x = \frac{q(t)}{3}\int_0^1 \partial_x(u_{\varepsilon n})^3\,\mathrm{d}x = 0,$$

$$\int_0^1 \partial_x^3 u_{\varepsilon n}u_{\varepsilon n}\,\mathrm{d}x = -\int_0^1 \partial_x^2 u_{\varepsilon n}\partial_x u_{\varepsilon n}\,\mathrm{d}x = \frac{1}{2}(\partial_x u(t,0))^2,$$

$$\int_0^1 r(t,x)\partial_x u_{\varepsilon n}u_{\varepsilon n}\,\mathrm{d}x = \int_0^1 r(t,x)\partial_x\left(\frac{u_{\varepsilon n}^2}{2}\right)\mathrm{d}x = -\frac{1}{2}\int_0^1 \partial_x r(t,x)u_{\varepsilon n}^2\,\mathrm{d}x.$$

Then

$$\frac{p(t)}{2}\frac{d}{dt}\int_0^1 u_{\varepsilon n}^2\,\mathrm{d}x + \varepsilon\int_0^1 (\partial_x^2 u_{\varepsilon n})^2\,\mathrm{d}x \le \frac{1}{2}\int_0^1 \partial_x r(t,x)u_{\varepsilon n}^2\,\mathrm{d}x + \int_0^1 f u_{\varepsilon n}\,\mathrm{d}x, \tag{4.18}$$

by integrating (4.18) with respect to t ($t\in(0,T)$), and according to (4.10), using Cauchy-Schwarz inequality, we find that

$$\begin{aligned}&\frac{\alpha}{2}\|u_{\varepsilon n}\|^2_{L^2(I)} + \varepsilon\int_0^t \|\partial_x^2 u_{\varepsilon n}(s)\|^2_{L^2(I)}\,\mathrm{d}s\\ &\le \int_0^t \|f(s)\|_{L^2(I)}\|u_{\varepsilon n}(s)\|_{L^2(I)}\,\mathrm{d}s + \frac{\beta}{2}\int_0^t \|u_{\varepsilon n}(s)\|^2_{L^2(I)}\,\mathrm{d}s.\end{aligned}$$

By the elementary inequality

$$|ab| \le \frac{\xi}{2}a^2 + \frac{b^2}{2\xi}, \quad \forall a,b\in\mathbb{R},\ \forall\xi>0, \tag{4.19}$$

with $\xi = 1$, we obtain

$$\begin{aligned}&\frac{\alpha}{2}\|u_{\varepsilon n}\|^2_{L^2(I)} + \frac{\varepsilon}{2}\int_0^t \|\partial_x^2 u_{\varepsilon n}(s)\|^2_{L^2(I)}\,\mathrm{d}s\\ &\le \frac{1}{2}\int_0^t \|f(s)\|^2_{L^2(I)}\,\mathrm{d}s + \frac{1}{2}\int_0^t \|u_{\varepsilon n}(s)\|^2_{L^2(I)}\,\mathrm{d}s + \frac{\beta}{2}\int_0^t \|u_{\varepsilon n}(s)\|^2_{L^2(I)}\,\mathrm{d}s.\end{aligned}$$

Hence

$$\alpha\|u_{\varepsilon n}\|^2_{L^2(I)} + \varepsilon \int_0^t \|\partial_x^2 u_{\varepsilon n}(s)\|^2_{L^2(I)}\, \mathrm{d}s$$
$$\le \int_0^t \|f(s)\|^2_{L^2(I)}\, \mathrm{d}s + (\beta+1) \int_0^t \|u_{\varepsilon n}(s)\|^2_{L^2(I)}\, \mathrm{d}s,$$

and

$$\alpha\|u_{\varepsilon n}\|^2_{L^2(I)} + \varepsilon \int_0^t \|\partial_x^2 u_{\varepsilon n}(s)\|^2_{L^2(I)}\, \mathrm{d}s$$
$$\le (\beta+1) \int_0^t \left(\|u_{\varepsilon n}(s)\|^2_{L^2(I)} + \varepsilon \int_0^s \|\partial_x^2 u_{\varepsilon n}(\tau)\|^2_{L^2(I)}\, \mathrm{d}\tau \right) \mathrm{d}s$$
$$+ \int_0^t \|f(s)\|^2_{L^2(I)}\, \mathrm{d}s.$$

As f is in $L^2(R)$, there exists a positive constant C_1 independent of n such that

$$\alpha\|u_{\varepsilon n}\|^2_{L^2(I)} + \varepsilon \int_0^t \|\partial_x^2 u_{\varepsilon n}(s)\|^2_{L^2(I)}\, \mathrm{d}s$$
$$\le C_1 + (\beta+1) \int_0^t \left(\|u_{\varepsilon n}(s)\|^2_{L^2(I)} + \varepsilon \int_0^s \|\partial_x^2 u_{\varepsilon n}(\tau)\|^2_{L^2(I)}\, \mathrm{d}\tau \right) \mathrm{d}s,$$

then, by Gronwall's inequality, we deduce that

$$\alpha\|u_{\varepsilon n}\|^2_{L^2(I)} + \varepsilon \int_0^t \|\partial_x^2 u_{\varepsilon n}(s)\|^2_{L^2(I)}\, \mathrm{d}s \le C_1 \exp((\beta+1)t).$$

Taking $K_1 = C_1 \exp((\beta+1)T)$, we obtain

$$\alpha\|u_{\varepsilon n}\|^2_{L^2(I)} + \varepsilon \int_0^t \|\partial_x^2 u_{\varepsilon n}(s)\|^2_{L^2(I)}\, \mathrm{d}s \le K_1.$$

□

Lemma 4.7. *There exists a positive constant $K_2(\varepsilon)$ independent of n, such that for all $t \in [0,T]$*

$$\alpha \|\partial_x^2 u_{\varepsilon n}\|^2_{L^2(I)} + \varepsilon \int_0^t \|\partial_x^4 u_{\varepsilon n}(s)\|^2_{L^2(I)} \,\mathrm{d}s \leq K_2(\varepsilon),$$

where α is given in (4.10).

Proof. As $\partial_x^4 e_j = \lambda_j e_j$, we have

$$\sum_{j=1}^n c_j(t)\lambda_j e_j = \sum_{j=1}^n c_j(t)\partial_x^4 e_j = \partial_x^4 u_{\varepsilon n}(t).$$

Multiplying both sides of (4.16) by $c_j\lambda_j$, taking the summation with respect to j from 1 to n and using the previous relation, we obtain

$$\begin{aligned}
&\frac{p(t)}{2}\frac{d}{dt}\int_0^1 (\partial_x^2 u_{\varepsilon n})^2 \,\mathrm{d}x + \varepsilon \int_0^1 (\partial_x^4 u_{\varepsilon n})^2 \,\mathrm{d}x \\
&= -q(t)\int_0^1 u_{\varepsilon n}\partial_x u_{\varepsilon n}\partial_x^4 u_{\varepsilon n} \,\mathrm{d}x - \int_0^1 \partial_x^3 u_{\varepsilon n}\partial_x^4 u_{\varepsilon n} \,\mathrm{d}x \qquad (4.20)\\
&- \int_0^1 r(t,x)\partial_x u_{\varepsilon n}\partial_x^4 u_{\varepsilon n} \,\mathrm{d}x + \varepsilon \int_0^1 \partial_x^2 u_{\varepsilon n}\partial_x^4 u_{\varepsilon n} \,\mathrm{d}x + \int_0^1 f\partial_x^4 u_{\varepsilon n} \,\mathrm{d}x.
\end{aligned}$$

Using (4.10) and (4.19) with $\xi = \dfrac{3}{\varepsilon}$, we obtain

$$\left|\int_0^1 \partial_x^3 u_{\varepsilon n}\partial_x^4 u_{\varepsilon n} \,\mathrm{d}x\right| \leq \frac{3}{2\varepsilon}\|\partial_x^3 u_{\varepsilon n}\|^2_{L^2(I)} + \frac{\varepsilon}{6}\|\partial_x^4 u_{\varepsilon n}\|^2_{L^2(I)}, \qquad (4.21)$$

$$\left|\int_0^1 f\partial_x^4 u_{\varepsilon n} \,\mathrm{d}x\right| \leq \frac{3}{2\varepsilon}\|f\|^2_{L^2(I)} + \frac{\varepsilon}{6}\|\partial_x^4 u_{\varepsilon n}\|^2_{L^2(I)}, \qquad (4.22)$$

$$\left|\int_0^1 r(t,x)\partial_x u_{\varepsilon n}\partial_x^4 u_{\varepsilon n} \,\mathrm{d}x\right| \leq \frac{3\beta^2}{2\varepsilon}\|\partial_x u_{\varepsilon n}\|^2_{L^2(I)} + \frac{\varepsilon}{6}\|\partial_x^4 u_{\varepsilon n}\|^2_{L^2(I)}. \qquad (4.23)$$

Now (4.19) with $\xi = 3$, gives

$$\varepsilon \int_0^1 \partial_x^2 u_{\varepsilon n}\partial_x^4 u_{\varepsilon n} \,\mathrm{d}x \leq \frac{3\varepsilon}{2}\int_0^1 (\partial_x^2 u_{\varepsilon n})^2 \,\mathrm{d}x + \frac{\varepsilon}{6}\int_0^1 (\partial_x^4 u_{\varepsilon n})^2 \,\mathrm{d}x. \qquad (4.24)$$

By the Ehrling inequality, there exists a positive constant $\eta(\varepsilon,\beta)$, such that

$$\|\partial_x^3 u_{\varepsilon n}\|^2_{L^2(I)} \le \frac{\varepsilon^2}{9}\|\partial_x^4 u_{\varepsilon n}\|^2_{L^2(I)} + \eta(\varepsilon)\,\|u_{\varepsilon n}\|^2_{L^2(I)}\,.$$

Using the previous inequality, (4.21) becomes

$$\left|\int_0^1 \partial_x^3 u_{\varepsilon n}\partial_x^4 u_{\varepsilon n}\,\mathrm{d}x\right| \le \frac{\varepsilon}{3}\|\partial_x^4 u_{\varepsilon n}\|^2_{L^2(I)} + C_2\|u_{\varepsilon n}\|^2_{L^2(I)}. \tag{4.25}$$

where $C_2 = \eta(\varepsilon)\dfrac{3}{2\varepsilon}$.

Now, we have to estimate the first term of the right-hand side of (4.20). An integration by parts gives

$$\begin{aligned}\int_0^1 u_{\varepsilon n}\partial_x u_{\varepsilon n}\partial_x^4 u_{\varepsilon n}\,\mathrm{d}x &= -\int_0^1 \partial_x(u_{\varepsilon n}\partial_x u_{\varepsilon n})\partial_x^3 u_{\varepsilon n}\,\mathrm{d}x\\ &= -\int_0^1 (\partial_x u_{\varepsilon n})^2\partial_x^3 u_{\varepsilon n}\,\mathrm{d}x - \int_0^1 u_{\varepsilon n}\partial_x^2 u_{\varepsilon n}\partial_x^3 u_{\varepsilon n}\,\mathrm{d}x\\ &= \int_0^1 \partial_x(\partial_x u_{\varepsilon n})^2\partial_x^2 u_{\varepsilon n}\,\mathrm{d}x - \frac{1}{2}\int_0^1 u_{\varepsilon n}\partial_x(\partial_x^2 u_{\varepsilon n})^2\,\mathrm{d}x\\ &= \frac{5}{2}\int_0^1 \partial_x u_{\varepsilon n}(\partial_x^2 u_{\varepsilon n})^2\,\mathrm{d}x.\end{aligned}$$

Then

$$\left|q(t)\int_0^1 u_{\varepsilon n}\partial_x u_{\varepsilon n}\partial_x^4 u_{\varepsilon n}\,\mathrm{d}x\right| \le \frac{5\beta}{2}\|\partial_x u_{\varepsilon n}\|_{L^\infty(I)}\|\partial_x^2 u_{\varepsilon n}\|^2_{L^2(I)}. \tag{4.26}$$

By integrating (4.20) between 0 and t, the estimates (4.22), (4.23), (4.24), (4.25) and (4.26) imply

$$\begin{aligned}&\frac{\alpha}{2}\|\partial_x^2 u_{\varepsilon n}\|^2_{L^2(I)} + \frac{\varepsilon}{6}\int_0^t \|\partial_x^4 u_{\varepsilon n}(s)\|^2_{L^2(I)}\,\mathrm{d}s\\ &\le \frac{3}{2\varepsilon}\int_0^t \|f(s)\|^2_{L^2(I)}\,\mathrm{d}s + \frac{5\beta}{2}\int_0^t \|\partial_x u_{\varepsilon n}\|_{L^\infty(I)}\|\partial_x^2 u_{\varepsilon n}\|^2_{L^2(I)}\,\mathrm{d}s\\ &+ \frac{3\varepsilon}{2}\int_0^t \|\partial_x^2 u_{\varepsilon n}\|^2_{L^2(I)}\,\mathrm{d}s + C_2\int_0^t \|u_{\varepsilon n}\|^2_{L^2(I)}\,\mathrm{d}s + \frac{3\beta^2}{2\varepsilon}\int_0^t \|\partial_x u_{\varepsilon n}\|^2_{L^2(I)}\,\mathrm{d}s.\end{aligned}$$

Using the injection of $H_0^1(I)$ in $L^\infty(I)$ and Poincaré's inequality, we obtain

$$\frac{\alpha}{2}\|\partial_x^2 u_{\varepsilon n}\|_{L^2(I)}^2+\frac{\varepsilon}{6}\int_0^t \|\partial_x^4 u_{\varepsilon n}(s)\|_{L^2(I)}^2\,\mathrm{d}s$$

$$\leq \int_0^t \|\partial_x^2 u_{\varepsilon n}(s)\|_{L^2(I)}^2\left(\frac{5\beta}{2}\|u_{\varepsilon n}(s)\|_{H^2(I)}^2+\frac{3\varepsilon}{2}+\frac{C_2}{4}+\frac{3\beta^2}{4\varepsilon}\right)\mathrm{d}s$$

$$+\frac{3}{2\varepsilon}\int_0^t \|f(s)\|_{L^2(I)}^2\,\mathrm{d}s.$$

Observe that $f\in L^2(R)$. Then, from (4.17), there exists a positive constant $C_3(\varepsilon)$ such that

$$\alpha\|\partial_x^2 u_{\varepsilon n}\|_{L^2(I)}^2+\varepsilon\int_0^t \|\partial_x^4 u_{\varepsilon n}(s)\|_{L^2(I)}^2\,\mathrm{d}s$$

$$\leq C_3(\varepsilon)+\int_0^t\left(\frac{5\beta}{2}\|u_{\varepsilon n}(s)\|_{H^2(I)}^2+\frac{3\varepsilon}{2}+\frac{C_2}{4}+\frac{3\beta^2}{4\varepsilon}\right)\|\partial_x^2 u_{\varepsilon n}\|_{L^2(I)}^2\,\mathrm{d}s,$$

so,

$$\|\partial_x^2 u_{\varepsilon n}\|_{L^2(I)}^2+\varepsilon\int_0^t \|\partial_x^4 u_{\varepsilon n}(s)\|_{L^2(I)}^2\,\mathrm{d}s$$

$$\leq C_3(\varepsilon)+\int_0^t\left(\frac{5\beta}{2}\|u_{\varepsilon n}(s)\|_{H^2(I)}^2+C_4(\varepsilon)\right)$$

$$\left(\|\partial_x^2 u_{\varepsilon n}\|_{L^2(I)}^2+\varepsilon\int_0^s \|\partial_x^4 u_{\varepsilon n}(\tau)\|_{L^2(I)}^2\,\mathrm{d}\tau\right)\mathrm{d}s.$$

where $C_4(\varepsilon)=\dfrac{3\varepsilon}{2}+\dfrac{C_2}{4}+\dfrac{3\beta^2}{4\varepsilon}$. Gronwall's inequality shows that

$$\alpha\|\partial_x^2 u_{\varepsilon n}\|_{L^2(I)}^2+\varepsilon\int_0^t \|\partial_x^4 u_{\varepsilon n}(s)\|_{L^2(I)}^2\,\mathrm{d}s$$

$$\leq C_3(\varepsilon)\exp\left(\int_0^t\left(\frac{5\beta}{2}\|u_{\varepsilon n}(s)\|_{H^2(I)}^2+C_4(\varepsilon)\right)\mathrm{d}s\right).$$

From (4.17), there exists a positive constant $K_2(\varepsilon)$ independent of n, such that

$$\alpha\|\partial_x^2 u_{\varepsilon n}\|^2_{L^2(I)} + \varepsilon \int_0^t \|\partial_x^4 u_{\varepsilon n}(s)\|^2_{L^2(I)}\,\mathrm{d}s \le K_2(\varepsilon).$$

□

Lemma 4.8. *There exists a positive constant $K_3(\varepsilon)$ independent of n, such that for all $t \in [0, T]$*

$$\alpha\|\partial_t u_{\varepsilon n}\|^2_{L^2(I)} + \varepsilon \int_0^t \|\left(\partial_t(\partial_x^2 u_{\varepsilon n}(s)\right)\|^2_{L^2(I)}\,\mathrm{d}s \le K_3(\varepsilon).$$

Proof. Differentiating (4.16) with respect to t, and replacing e_j by $\partial_t u_{\varepsilon n}$, we get

$$\begin{aligned}
&\int_0^1 \partial_t\left(p(t)\partial_t u_{\varepsilon n}\right)\partial_t u_{\varepsilon n}\,\mathrm{d}x + \int_0^1 \partial_t\left(q(t)u_{\varepsilon n}\partial_x u_{\varepsilon n}\right)\partial_t u_{\varepsilon n}\,\mathrm{d}x\\
&+ \int_0^1 \partial_t\left(\partial_x^3 u_{\varepsilon n}\right)\partial_t u_{\varepsilon n}\,\mathrm{d}x + \int_0^1 \partial_t\left(r(t,x)\partial_x u_{\varepsilon n}\right)\partial_t u_{\varepsilon n}\,\mathrm{d}x\\
&- \varepsilon\int_0^1 \partial_t\left(\partial_x^2 u_{\varepsilon n}\right)\partial_t u_{\varepsilon n}\,\mathrm{d}x + \varepsilon\int_0^1 \partial_t\left(\partial_x^4 u_{\varepsilon n}\right)\partial_t u_{\varepsilon n}\,\mathrm{d}x\\
&= \int_0^1 \partial_t f \partial_t u_{\varepsilon n}\,\mathrm{d}x.
\end{aligned}$$

By the boundary conditions of (4.14), and (4.10) we obtain

$$\begin{aligned}
&\alpha\|\partial_t u_{\varepsilon n}\|^2_{L^2(I)} + \varepsilon \int_0^t \|\left(\partial_t(\partial_x^2 u_{\varepsilon n}(s)\right)\|^2_{L^2(I)}\,\mathrm{d}s\\
&\le \frac{1}{2}\|\partial_t f\|^2_{L^2(R)} + \frac{\beta}{2}\|u_{\varepsilon n}\|_{L^\infty(I)}\|\partial_x u_{\varepsilon n}\|^2_{L^2(R)} + \beta\|\partial_x u_{\varepsilon n}\|^2_{L^2(R)}\\
&+ \left(\frac{1}{2} + \frac{3\beta}{2}\|u_{\varepsilon n}\|^2_{L^\infty(R)} + 3\beta\|\partial_x u_{\varepsilon n}\|^2_{L^\infty(R)}\right)\|\partial_t u_{\varepsilon n}\|^2_{L^2(R)}.
\end{aligned}$$

Then, Gronwall's inequality shows that

$$\alpha\|\partial_t u_{\varepsilon n}\|^2_{L^2(I)} + \varepsilon \int_0^t \|\left(\partial_t(\partial_x^2 u_{\varepsilon n}(s)\right)\|^2_{L^2(I)}\,\mathrm{d}s \le K_3(\varepsilon).$$

□

4.3.2 Existence of solution for the approximated problem

Lemmas 4.6, 4.7 and 4.8 show that the Galerkin approximation $u_{\varepsilon n}$ is bounded in $L^\infty(0,T,H^2(I))$, and $\varepsilon\partial_x^4 u_{\varepsilon n}$, $\partial_t u_{\varepsilon n}$ are bounded in $L^2(R)$.

So, it is possible to extract a subsequence from $u_{\varepsilon n}$, still denoted $u_{\varepsilon n}$, such that

$$u_{\varepsilon n} \to u_\varepsilon \quad \text{weakly in} \quad L^2\left(0,T,H_0^1(I)\right), \tag{4.27}$$

$$u_{\varepsilon n} \to u_\varepsilon \quad \text{strongly in} \quad L^2\left(0,T,L^2(I)\right), \tag{4.28}$$

$$\varepsilon\partial_x^4 u_{\varepsilon n} \to \varepsilon\partial_x^4 u_\varepsilon \quad \text{weakly in} \quad L^2\left(0,T,L^2(I)\right), \tag{4.29}$$

$$\partial_t u_{\varepsilon n} \to \partial_t u_\varepsilon, \quad \text{weakly in} \quad L^2\left(0,T,L^2(I)\right). \tag{4.30}$$

Note that (4.30) implies

$$\int_0^T\int_0^1 p(t)\partial_t u_{\varepsilon n} w \,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_0^T\int_0^1 p(t)\partial_t u_\varepsilon w \,\mathrm{d}x\,\mathrm{d}t, \qquad \forall w \in L^2(R).$$

From (4.27) and (4.28)

$$u_{\varepsilon n}\partial_x u_{\varepsilon n} \longrightarrow u_\varepsilon \partial_x u_\varepsilon \quad \text{weakly in} \quad L^2(R),$$

then

$$\int_0^T\int_0^1 q(t) u_{\varepsilon n}\partial_x u_{\varepsilon n} w \,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_0^T\int_0^1 q(t) u_\varepsilon\partial_x u_\varepsilon w \,\mathrm{d}x\,\mathrm{d}t, \qquad \forall w \in L^2(R).$$

$$\int_0^T\int_0^1 \partial_x^3 u_{\varepsilon n} w \,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_0^T\int_0^1 \partial_x^3 u_\varepsilon w \,\mathrm{d}x\,\mathrm{d}t, \qquad \forall w \in L^2(R),$$

$$\int_0^T\int_0^1 r(t,x)\partial_x u_{\varepsilon n} w \,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_0^T\int_0^1 r(t,x)\partial_x u_\varepsilon w \,\mathrm{d}x\,\mathrm{d}t, \qquad \forall w \in L^2(R),$$

and

$$\varepsilon\int_0^T\int_0^1 (-\partial_x^2 u_{\varepsilon n} + \partial_x^4 u_{\varepsilon n}) w \,\mathrm{d}x\,\mathrm{d}t \longrightarrow \varepsilon\int_0^T\int_0^1 (-\partial_x^2 u_\varepsilon + \partial_x^4 u_\varepsilon) w \,\mathrm{d}x\,\mathrm{d}t, \qquad \forall w \in L^2(R).$$

Using these properties, we can pass to the limit in (4.16) as $n \longrightarrow +\infty$. Consequently, there exists a solution u_ε of (4.14).

4.4 Passage to the limit in ε

Our goal in this section is to find uniform estimates in ε, to pass to the limit in (4.14) when $\varepsilon \longrightarrow 0$.

Lemma 4.9. *Under the hypotheses of Theorem 4.2 , for all $\varepsilon \in (0, \alpha)$, the solution of* (4.14) *satisfies the estimate*

$$\|u_\varepsilon\|^2_{L^2(I)} + (\alpha - \varepsilon) \int_0^t \|\partial_x u_\varepsilon(s)\|^2_{L^2(I)} \, \mathrm{d}s + \varepsilon \int_0^t \|\partial_x^2 u_\varepsilon(s)\|^2_{L^2(I)} \, \mathrm{d}s \leq K_4,$$

where α is given in (4.10) *and K_4 is a positive constant independent of ε.*

Proof. Multiplying both sides of (4.14) by u_ε and integrating between 0 and 1, we obtain

$$\frac{p(t)}{2} \frac{d}{dt} \int_0^1 u_\varepsilon^2 \, \mathrm{d}x + \varepsilon \int_0^1 (\partial_x u_\varepsilon)^2 \, \mathrm{d}x + \varepsilon \int_0^1 (\partial_x^2 u_\varepsilon)^2 \, \mathrm{d}x$$
$$\frac{1}{2}(\partial_x u(t,0))^2 - \frac{1}{2} \int_0^1 \partial_x r(t,x) u_\varepsilon^2 \, \mathrm{d}x = \int_0^1 f u_\varepsilon \, \mathrm{d}x.$$

Following the same steps as in lemma 4.6, we prove that there exists a constant K_1 independent of ε, such that

$$\alpha \|u_\varepsilon\|^2_{L^2(I)} + \varepsilon \int_0^t \|\partial_x^2 u_\varepsilon(s)\|^2_{L^2(I)} \, \mathrm{d}s \leq K_1. \tag{4.31}$$

Now, we multiply both sides of (4.14) by $e^x u_\varepsilon$ and integrate between 0 and 1, to get

$$\begin{aligned} & p(t) \int_0^1 e^x u_\varepsilon \partial_t u_\varepsilon \, \mathrm{d}x + \int_0^1 e^x u_\varepsilon \partial_x^3 u_\varepsilon \, \mathrm{d}x \\ & - \varepsilon \int_0^1 e^x u_\varepsilon \partial_x^2 u_\varepsilon \, \mathrm{d}x + \varepsilon \int_0^1 e^x u_\varepsilon \partial_x^4 u_\varepsilon \, \mathrm{d}x \\ & = -q(t) \int_0^1 e^x u_\varepsilon^2 \partial_x u_\varepsilon \, \mathrm{d}x - \int_0^1 r(t,x) e^x u_\varepsilon \partial_x u_\varepsilon \, \mathrm{d}x + \int_0^1 f e^x u_\varepsilon \, \mathrm{d}x. \end{aligned} \tag{4.32}$$

Taking into account the boundary conditions of Problem (4.14), an integration by parts gives

$$\int_0^1 e^x u_\varepsilon \partial_t u_\varepsilon \,\mathrm{d}x = \frac{1}{2}\frac{d}{dt}\int_0^1 e^x u_\varepsilon^2 \,\mathrm{d}x,$$

$$\begin{aligned}\int_0^1 e^x u \partial_x^3 u_\varepsilon \,\mathrm{d}x &= -\int_0^1 e^x u_\varepsilon \partial_x^2 u_\varepsilon \,\mathrm{d}x - \int_0^1 e^x \partial_x u \partial_x^2 u_\varepsilon \,\mathrm{d}x \\ &= \int_0^1 e^x u_\varepsilon \partial_x u_\varepsilon \,\mathrm{d}x + \int_0^1 e^x (\partial_x u_\varepsilon)^2 \,\mathrm{d}x - \frac{1}{2}\int_0^1 e^x \partial_x (\partial_x u_\varepsilon)^2 \,\mathrm{d}x \\ &= -\frac{1}{2}\int_0^1 e^x u_\varepsilon^2 \,\mathrm{d}x + \frac{3}{2}\int_0^1 e^x (\partial_x u_\varepsilon)^2 \,\mathrm{d}x + \frac{1}{2}(\partial_x u(t,0))^2,\end{aligned}$$

$$\begin{aligned}\int_0^1 e^x u_\varepsilon \partial_x^2 u_\varepsilon \,\mathrm{d}x &= -\int_0^1 e^x u_\varepsilon \partial_x u_\varepsilon \,\mathrm{d}x - \int_0^1 e^x (\partial_x u_\varepsilon)^2 \,\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 e^x u_\varepsilon^2 \,\mathrm{d}x - \int_0^1 e^x (\partial_x u_\varepsilon)^2 \,\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}\int_0^1 e^x u_\varepsilon \partial_x^4 u_\varepsilon \,\mathrm{d}x &= -\int_0^1 e^x u_\varepsilon \partial_x^3 u_\varepsilon \,\mathrm{d}x - \int_0^1 e^x \partial_x u_\varepsilon \partial_x^3 u_\varepsilon \,\mathrm{d}x \\ &= \int_0^1 e^x u_\varepsilon \partial_x^2 u_\varepsilon \,\mathrm{d}x + \int_0^1 e^x \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \,\mathrm{d}x + \int_0^1 e^x \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \,\mathrm{d}x + \int_0^1 e^x (\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x \\ &= -\int_0^1 e^x u_\varepsilon \partial_x u_\varepsilon \,\mathrm{d}x - \int_0^1 e^x (\partial_x u_\varepsilon)^2 \,\mathrm{d}x + \int_0^1 e^x \partial_x (\partial_x u_\varepsilon)^2 \,\mathrm{d}x + \int_0^1 e^x (\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 e^x u_\varepsilon^2 \,\mathrm{d}x - 2\int_0^1 e^x (\partial_x u_\varepsilon)^2 \,\mathrm{d}x + \int_0^1 e^x (\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x - (\partial_x u_\varepsilon(t,0))^2,\end{aligned}$$

$$\begin{aligned}\int_0^1 r(t,x)e^x u_\varepsilon \partial_x u_\varepsilon \, \mathrm{d}x &= \frac{1}{2}\int_0^1 r(t,x)e^x \partial_x (u_\varepsilon)^2 \, \mathrm{d}x \\ &= -\frac{1}{2}\int_0^1 \partial_x r(t,x) e^x u_\varepsilon^2 \, \mathrm{d}x - \frac{1}{2}\int_0^1 r(t,x)e^x u_\varepsilon^2 \, \mathrm{d}x,\end{aligned}$$

and

$$\int_0^1 e^x u_\varepsilon^2 \partial_x u_\varepsilon \, \mathrm{d}x = -\frac{1}{3}\int_0^1 e^x u_\varepsilon^3 \, \mathrm{d}x.$$

Using the previous calculations, (4.33) becomes

$$\begin{aligned}&\frac{p(t)}{2}\frac{d}{dt}\int_0^1 e^x u_\varepsilon^2 \, \mathrm{d}x + \left(\frac{3}{2} - \varepsilon\right)\int_0^1 e^x (\partial_x u_\varepsilon)^2 \, \mathrm{d}x \\ &+ (\frac{1}{2} - \varepsilon)(\partial_x u(t,0))^2 + \varepsilon \int_0^1 e^x (\partial_x^2 u_\varepsilon)^2 \, \mathrm{d}x \\ &= \int_0^1 \frac{e^x}{2}\left(\partial_x r(t,x) + r(t,x) +\right) u_\varepsilon^2 \, \mathrm{d}x + \frac{q(t)}{3}\int_0^1 e^x u_\varepsilon^3 \, \mathrm{d}x + \int_0^1 f e^x u_\varepsilon \, \mathrm{d}x.\end{aligned}$$

As $1 \le e^x \le e$ for all $x \in [0,1]$, by integrating from 0 at t ($t \in (0,T)$) and using (4.10), we find thanks to Cauchy-Schwarz inequality

$$\begin{aligned}&\alpha \|u_\varepsilon\|_{L^2(I)}^2 + (3\alpha - 2\varepsilon)\int_0^t \|\partial_x u_\varepsilon(s)\|_{L^2(I)}^2 \, \mathrm{d}s + \varepsilon \int_0^t \|\partial_x^2 u_\varepsilon(s)\|_{L^2(I)}^2 \, \mathrm{d}s \\ &\le e(3\beta + 2\varepsilon + 1)\int_0^t \|u_\varepsilon(s)\|_{L^2(I)}^2 \, \mathrm{d}s + e\int_0^t \|f(s)\|_{L^2(I)}^2 \, \mathrm{d}s + \frac{2e\beta}{3}\int_0^t \|u_\varepsilon\|_{L^3(I)}^3 \, \mathrm{d}s.\end{aligned}$$

Now, we will get an estimate for the last term in the previous inequality. Since

$$\|u_\varepsilon\|_{L^\infty(I)}^2 \le 2\|u_\varepsilon\|_{L^2(I)}\|\partial_x u_\varepsilon\|_{L^2(I)},$$

then

$$\begin{aligned}\frac{2e\beta}{3}\|u_\varepsilon\|_{L^3(I)}^3 &\le \frac{2e\beta}{3}\|u_\varepsilon\|_{L^\infty(I)}\|u_\varepsilon\|_{L^2(I)}^2 \\ &\le e\beta \|\partial_x u_\varepsilon\|_{L^2(I)}^{\frac{1}{2}}\|u_\varepsilon\|_{L^2(I)}^{\frac{5}{2}}.\end{aligned}$$

By Young's inequality $|AB| \leq \dfrac{|A|^p}{p} + \dfrac{|B|^{p'}}{p'}$ with $1 < p < \infty$ and $p' = \dfrac{p}{p-1}$, choosing $p = 4$ (then $p' = \dfrac{4}{3}$)

$$A = \sqrt{2}\alpha^{\frac{1}{4}} \|\partial_x u_\varepsilon\|_{L^2(I)}^{\frac{1}{2}},$$
$$B = \frac{e\beta}{\sqrt{2}\alpha^{\frac{1}{4}}} \|u_\varepsilon\|_{L^2(I)}^{\frac{5}{2}},$$

we obtain

$$e\beta \|\partial_x u_\varepsilon\|_{L^2(I)}^{\frac{1}{2}} \|u_\varepsilon\|_{L^2(I)}^{\frac{5}{2}} \leq \alpha \|\partial_x u_\varepsilon\|_{L^2(I)}^2 + \frac{3}{4}\left(\frac{e\beta}{\sqrt{2}\alpha^{\frac{1}{4}}}\right)^{\frac{4}{3}} \|u_\varepsilon\|_{L^2(I)}^{\frac{10}{3}}.$$

On the other hand, $f \in L^2(R)$, then there exists a constant C_5 independent of ε such that

$$\begin{aligned}
&\alpha \|u_\varepsilon\|_{L^2(I)}^2 + 2(\alpha - \varepsilon) \int_0^t \|\partial_x u_\varepsilon(s)\|_{L^2(I)}^2 \, \mathrm{d}s + \varepsilon \int_0^t \|\partial_x^2 u_\varepsilon(s)\|_{L^2(I)}^2 \, \mathrm{d}s \\
&\leq C_5 + e(3\beta + 2\varepsilon + 1) \int_0^t \|u_\varepsilon(s)\|_{L^2(I)}^2 \, \mathrm{d}s \\
&+ \frac{3}{4}\left(\frac{e\beta}{\sqrt{2}\alpha^{\frac{1}{4}}}\right)^{\frac{4}{3}} \int_0^t \|u_\varepsilon\|_{L^2(I)}^{\frac{4}{3}} \|u_\varepsilon\|_{L^2(I)}^2 \, \mathrm{d}s.
\end{aligned}$$

As ε goes to 0 we can choose $\varepsilon \in (0, \alpha)$. Then, we deduce that

$$\varphi(t) = \alpha \|u_\varepsilon\|_{L^2(I)}^2 + (\alpha - \varepsilon) \int_0^t \|\partial_x u_\varepsilon(s)\|_{L^2(I)}^2 \, \mathrm{d}s + \varepsilon \int_0^t \|\partial_x^2 u_\varepsilon(s)\|_{L^2(I)}^2 \, \mathrm{d}s$$

verifies the following inequality

$$\varphi(t) \leq C_5 + \int_0^t \left(C_6 \|u_\varepsilon\|_{L^2(I)}^{\frac{4}{3}} + C_7\right) \varphi(s) \, \mathrm{d}s,$$

where $C_6 = \dfrac{3}{4}\left(\dfrac{e\beta}{\sqrt{2}\alpha^{\frac{1}{4}}}\right)^{\frac{4}{3}}$ and $C_7 = e(3\beta + 2\varepsilon + 1)$.

By the Gronwall's inequality, we obtain

$$\varphi(t) \leq C_5 \exp\left(\int_0^t C_6 \|\partial_x u_\varepsilon\|_{L^2(I)}^{\frac{4}{3}} \, \mathrm{d}s + C_7 T\right).$$

From (4.31) we have a bound for $\int\limits_0^t \|u_\varepsilon\|_{L^2(I)}^{4/3}\mathrm{d}s$, then there exists a positive constant K_4 independent of ε such that

$$\|u_\varepsilon\|_{L^2(I)}^2 + (\alpha-\varepsilon)\int\limits_0^t \|\partial_x u_\varepsilon(s)\|_{L^2(I)}^2\,\mathrm{d}s + \varepsilon\int\limits_0^t \|\partial_x^2 u_\varepsilon(s)\|_{L^2(I)}^2\,\mathrm{d}s \le K_4$$

□

Lemma 4.10. *Under the hypotheses of Theorem 4.2 , for all $\varepsilon > 0$, the solution of* (4.14) *satisfies the estimate*

$$\alpha\|\partial_t u_\varepsilon\|_{L^2(I)}^2 + C\int\limits_0^t \|\partial_t(\partial_x u_\varepsilon(s))\|_{L^2(I)}^2\,\mathrm{d}s + \varepsilon\int\limits_0^t \|\partial_t(\partial_x^2 u_\varepsilon(s))\|_{L^2(I)}^2\,\mathrm{d}s \le K_5$$

where C and K_5 are a positive constants independent of ε.

Proof. Differentiating (4.14) with respect to t, multiplying both sides by $e^x\partial_t u_\varepsilon$ and integrating between 0 and 1, we obtain

$$\begin{aligned}
&p(t)\int\limits_0^1 e^x\partial_t u_\varepsilon \partial_t^2 u_\varepsilon\,\mathrm{d}x + p'(t)\int\limits_0^1 e^x(\partial_t u_\varepsilon)^2\,\mathrm{d}x + \int\limits_0^1 e^x\partial_t\left(q(t)u_\varepsilon\partial_x u_\varepsilon\right)\partial_t u_\varepsilon\,\mathrm{d}x\\
&+\int\limits_0^1 e^x\partial_t\left(\partial_x^3 u_\varepsilon\right)\partial_t u_\varepsilon\,\mathrm{d}x + \int\limits_0^1 e^x\partial_t\left(r(t,x)\partial_x u_\varepsilon\right)\partial_t u_\varepsilon\,\mathrm{d}x\\
&-\varepsilon\int\limits_0^1 e^x\partial_t\left(\partial_x^2 u_\varepsilon\right)\partial_t u_\varepsilon\,\mathrm{d}x + \varepsilon\int\limits_0^1 e^x\partial_t\left(\partial_x^4 u_\varepsilon\right)\partial_t u_\varepsilon\,\mathrm{d}x\\
&=\int\limits_0^1 e^x\partial_t f\partial_t u_\varepsilon\,\mathrm{d}x.
\end{aligned}\tag{4.33}$$

By the boundary conditions of (4.14), an integration by parts gives

$$\int\limits_0^1 e^x\partial_t u_\varepsilon\partial_t^2 u_\varepsilon\,\mathrm{d}x = \frac{1}{2}\frac{d}{dt}\int\limits_0^1 e^x(\partial_t u_\varepsilon)^2\,\mathrm{d}x,$$

$$\int_0^1 e^x \partial_t \left(\partial_x^2 u_\varepsilon\right) \partial_t u_\varepsilon \,\mathrm{d}x = \frac{1}{2}\int_0^1 e^x (\partial_t u_\varepsilon)^2 \,\mathrm{d}x - \int_0^1 e^x (\partial_t(\partial_x u_\varepsilon))^2 \,\mathrm{d}x,$$

$$\int_0^1 e^x \partial_t \left(\partial_x^4 u_\varepsilon\right) \partial_t u_\varepsilon \,\mathrm{d}x = \frac{1}{2}\int_0^1 e^x (\partial_t u_\varepsilon)^2 \,\mathrm{d}x - 2\int_0^1 e^x (\partial_t(\partial_x u_\varepsilon))^2 \,\mathrm{d}x + \int_0^1 e^x (\partial_t(\partial_x^2 u_\varepsilon))^2 \,\mathrm{d}x - (\partial_t(\partial_x u_\varepsilon(t,0)))^2,$$

$$\int_0^1 e^x \partial_t \left(r(t,x)\partial_x u_\varepsilon\right) \partial_t u_\varepsilon \,\mathrm{d}x = \int_0^1 e^x \partial_t r(t,x)\partial_x u_\varepsilon \partial_t u_\varepsilon \,\mathrm{d}x - \frac{1}{2}\int_0^1 \partial_x r(t,x) e^x (\partial_t u_\varepsilon)^2 \,\mathrm{d}x - \frac{1}{2}\int_0^1 r(t,x) e^x (\partial_t u_\varepsilon)^2 \,\mathrm{d}x,$$

and

$$\int_0^1 e^x \partial_t \left(\partial_x^3 u_\varepsilon\right) \partial_t u_\varepsilon \,\mathrm{d}x = -\frac{1}{2}\int_0^1 e^x (\partial_t u_\varepsilon)^2 \,\mathrm{d}x + \frac{3}{2}\int_0^1 e^x (\partial_t(\partial_x u_\varepsilon))^2 \,\mathrm{d}x + \frac{1}{2}(\partial_t(\partial_x u_\varepsilon(t,0)))^2.$$

Using the previous calculations, (4.33) becomes

$$\begin{aligned}
&\frac{p(t)}{2}\frac{d}{dt}\int_0^1 e^x (\partial_t u_\varepsilon)^2 \,\mathrm{d}x + \left(\frac{3}{2} - \varepsilon\right)\int_0^1 e^x (\partial_t(\partial_x u_\varepsilon))^2 \,\mathrm{d}x \\
&+ \left(\frac{1}{2} - \varepsilon\right)(\partial_t(\partial_x u(t,0)))^2 + \varepsilon \int_0^1 e^x (\partial_t(\partial_x^2 u_\varepsilon))^2 \,\mathrm{d}x \\
&= \int_0^1 \frac{e^x}{2}\left(\partial_x r(t,x) + r(t,x) - p'(t) + 1\right)(\partial_t u_\varepsilon)^2 \,\mathrm{d}x + \int_0^1 e^x \partial_t u_\varepsilon \partial_t f \,\mathrm{d}x \\
&+ q'(t)\int_0^1 e^x u_\varepsilon \partial_x u_\varepsilon \partial_t u_\varepsilon \,\mathrm{d}x + \int_0^1 e^x \partial_t r(t,x)\partial_x u_\varepsilon \partial_t u_\varepsilon \,\mathrm{d}x \\
&+ q(t)\left(\int_0^1 e^x \partial_x u_\varepsilon (\partial_t u_\varepsilon)^2 \,\mathrm{d}x + \int_0^1 e^x u_\varepsilon \partial_t u_\varepsilon \partial_t(\partial_x u_\varepsilon) \,\mathrm{d}x\right).
\end{aligned}$$

Moreover, we have

$$\left|\int_0^1 e^x \partial_t r(t,x)\partial_x u_\varepsilon \partial_t u_\varepsilon \,\mathrm{d}x\right| \leq \frac{\beta e}{2}\|\partial_x u_\varepsilon\|^2_{L^2(I)} + \frac{\beta e}{2}\|\partial_t u_\varepsilon\|^2_{L^2(I)}$$

$$\left|q'(t)\int_0^1 e^x u_\varepsilon \partial_x u_\varepsilon \partial_t u_\varepsilon \,\mathrm{d}x\right| \leq \frac{\beta e}{2}\|u_\varepsilon\|_{L^\infty(I)}\left(\|u_\varepsilon\|^2_{L^2(I)} + \|\partial_t u_\varepsilon\|^2_{L^2(I)}\right),$$

$$\left|q(t)\int_0^1 e^x u_\varepsilon (\partial_t u_\varepsilon)^2 \,\mathrm{d}x\right| \leq \beta e\|\partial_x u_\varepsilon\|_{L^2(I)}\|\partial_t u_\varepsilon\|^2_{L^2(I)}$$

$$\left|q(t)\int_0^1 e^x u_\varepsilon \partial_t u_\varepsilon \partial_t(\partial_x u_\varepsilon) \,\mathrm{d}x)\right| \leq \frac{\beta e}{2}\|u_\varepsilon\|_{L^\infty(I)}\left(\|\partial_t u_\varepsilon\|^2_{L^2(I)} + \|\partial_t(\partial_x u_\varepsilon)\|^2_{L^2(I)}\right)$$

Then, by the Gronwall's inequality we prove that there exist two positive constants K_5 and C independent of ε, such that

$$\alpha\|\partial_t u_\varepsilon\|^2_{L^2(I)} + C\int_0^t \|\partial_t(\partial_x u_\varepsilon(s))\|^2_{L^2(I)}\,\mathrm{d}s + \varepsilon\int_0^t \|\partial_t(\partial_x^2 u_\varepsilon(s))\|^2_{L^2(I)}\,\mathrm{d}s \leq K_5$$

□

Lemma 4.11. *Under the hypotheses of Theorem 4.2 ,* (4.8) *admits a solution u.*

Proof. By lemmas 4.9 and 4.10 it is possible to extract a subsequence from u_ε, still denoted u_ε, such that

$$u_\varepsilon \to u, \quad \text{weakly in} \quad L^2\left(0,T,H_0^1(I)\right), \tag{4.34}$$

$$u_\varepsilon\partial_x u_\varepsilon \to u\partial_x u, \quad \text{weakly in} \quad L^2\left(0,T,L^2(I)\right), \tag{4.35}$$

$$\varepsilon\partial_x^2 u_\varepsilon \to 0, \quad \text{weakly in} \quad L^2\left(0,T,L^2(I)\right). \tag{4.36}$$

$$\partial_t u_\varepsilon \to \partial_t u, \quad \text{weakly in} \quad L^2\left(0,T,L^2(I)\right). \tag{4.37}$$

From (4.15), for all $w \in L^2(0,T,H^2(I)\cap H_0^1(I))$ with $\partial_x w(t,0)=0$, we have

$$p(t)\int_0^T\int_0^1 \partial_t u_\varepsilon w \,\mathrm{d}x\,\mathrm{d}t + q(t)\int_0^T\int_0^1 u_\varepsilon \partial_x u_\varepsilon w \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_0^1 \partial_x u_\varepsilon \partial_x^2 w \,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_0^1 r(t,x)\partial_x u_\varepsilon w \,\mathrm{d}x\,\mathrm{d}t - \varepsilon\int_0^T\int_0^1 \partial_x^2 u_\varepsilon w \,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_0^T\int_0^1 \partial_x^2 u_\varepsilon \partial_x^2 w \,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_0^T\int_0^1 f w \,\mathrm{d}x\,\mathrm{d}t.$$

By (4.34)-(4.37), we can pass to the limit as $\varepsilon \longrightarrow 0$ and obtain

$$p(t)\int_0^T\int_0^1 \partial_t u w \,\mathrm{d}x\,\mathrm{d}t + q(t)\int_0^T\int_0^1 u_\varepsilon \partial_x u w \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_0^1 \partial_x u \partial_x^2 w \,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_0^1 r(t,x)\partial_x u w \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_0^1 f w \,\mathrm{d}x\,\mathrm{d}t.$$

This completes the proof of existence part for Theorem 4.2 . □

Lemma 4.12. *Under the hypotheses of Theorem 4.2 , the solution of Problem* (4.8) *is unique.*

Proof. Let u_1, u_2 be two solutions of (4.8) and $u = u_1 - u_2$. The equations satisfied by u_1 and u_2 lead to

$$\begin{aligned} &p(t)\partial_t u(t,x) + q(t)u(t,x)\partial_x u_1(t,x) + q(t)u_2(t,x)\partial_x u(t,x)\\ &+\partial_x^3 u(t,x) + r(t,x)\partial_x u(t,x) = 0. \end{aligned} \tag{4.38}$$

Multiplying both sides of (4.38) by $e^x u$ and integrating from 0 to 1 (with respect to x), we obtain

$$p(t)\int_0^1 e^x u\partial_t u \,\mathrm{d}x + q(t)\int_0^1 e^x u^2 \partial_x u_1 \,\mathrm{d}x + q(t)\int_0^1 e^x u_2 u \partial_x u \,\mathrm{d}x$$
$$+\int_0^1 e^x \partial_x^3 u u \,\mathrm{d}x\,\mathrm{d}x + \int_0^1 e^x r(t,x)\partial_x u u \,\mathrm{d}x = 0.$$

Taking into account the boundary conditions of Problem (4.8), an integration by parts gives because

$$q(t)\int_0^1 e^x u^2 \partial_x u_1 \,\mathrm{d}x = -q(t)\int_0^1 e^x u^2 u_1 \,\mathrm{d}x - 2q(t)\int_0^1 e^x u \partial_x u u_1 \,\mathrm{d}x$$

$$\int_0^1 e^x u \partial_x^3 u \,\mathrm{d}x = 2\int_0^1 e^x (\partial_x u)^2 \,\mathrm{d}x + \frac{1}{2}(\partial_x u(t,0))^2$$

$$\int_0^1 r(t,x) e^x u \partial_x u \,\mathrm{d}x = -\frac{1}{2}\int_0^1 \partial_x r(t,x) e^x u^2 \,\mathrm{d}x - \frac{1}{2}\int_0^1 r(t,x) e^x u^2 \,\mathrm{d}x$$

So,

$$\frac{p(t)}{2}\frac{d}{dt}\|e^x u\|^2_{L^2(I)} + 2\alpha\|\partial_x u\|^2_{L^2(I)} + \frac{1}{2}(\partial_x u(t,0))^2$$
$$= q(t)\int_0^1 e^x u^2 u_1 \,\mathrm{d}x + 2q(t)\int_0^1 e^x u_1 u \partial_x u \,\mathrm{d}x - q(t)\int_0^1 e^x u_2 u \partial_x u \,\mathrm{d}x$$
$$-\frac{1}{2}\int_0^1 \partial_x r(t,x) e^x u^2 \,\mathrm{d}x - \frac{1}{2}\int_0^1 r(t,x) e^x u^2 \,\mathrm{d}x.$$

By (4.19) with $\xi = 2\alpha$, we obtain

$$\left|\int_0^1 q(t)\left(2u_1 - u_2\right) e^x u \partial_x u \,\mathrm{d}x\right|$$
$$\leq e\beta\left[\left(2\|u_1\|_{L^\infty(R)} + \|u_2\|_{L^\infty(R)}\right)\|u\|_{L^2(I)}\right]\|\partial_x u\|_{L^2(I)}$$
$$\leq \frac{e^2\beta^2}{4\alpha}\left(2\|u_1\|_{L^\infty(R)} + \|u_2\|_{L^\infty(R)}\right)^2\|u\|^2_{L^2(I)} + \alpha\|\partial_x u\|^2_{L^2(I)},$$

and

$$\left|-\frac{1}{2}\int_0^1 \partial_x r(t,x) e^x u^2 \,\mathrm{d}x - \frac{1}{2}\int_0^1 r(t,x) e^x u^2 \,\mathrm{d}x\right| \leq \beta e\|u\|^2_{L^2(I)}.$$

Finally, we deduce that there exists a positive constant K, such that

$$\frac{d}{dt}\|u\|^2_{L^2(I)} \leq K\|u\|_{L^2(I)}.$$

Gronwall's lemma leads to $u = 0$. This completes the proof of Theorem 4.2 . □