Chapter 3

Existence of solution to Burgers equation in a non-parabolic domain

In this chapter, we study the Burgers equation with time variable coefficients, subject to boundary condition in a non-parabolic domain. Some assumptions on the boundary of the domain and on the coefficients of the equation will be imposed. The right-hand side of the equation is taken in L^2. The method we used is based on the approximation of the non-parabolic domain by a sequence of subdomains which can be transformed into regular domains. This work is an extension of the Burgers problem in domain that can be transformed into rectangle presented in Chapter 2.

3.1 Introduction

We consider the boundary value problem for the non homogeneous Burgers equation

$$\begin{cases} \partial_t u(t,x) + c(t)u(t,x)\partial_x u(t,x) - \partial_x^2 u(t,x) = f(t,x) & (t,x) \in \Omega, \\ u(t,\varphi_1(t)) = u(t,\varphi_2(t)) = 0 & t \in (0,T), \end{cases} \tag{3.1}$$

in $\Omega \subset \mathbb{R}^2$, where

$$\Omega = \left\{(t,x) \in \mathbb{R}^2;\ 0 < t < T,\ x \in I_t\right\},$$

$$I_t = \{x \in \mathbb{R};\ \varphi_1(t) < x < \varphi_2(t),\ t \in]0,T[\},$$

with

$$\varphi_1(0) = \varphi_2(0). \tag{3.2}$$

T is a positive number, $f \in L^2(\Omega)$ and $c(t)$ is given. The functions φ_1, φ_2 are defined on $[0,T]$, and belong to $\mathcal{C}^1(0,T)$.

The most interesting point of the problem studied in this chapter is the fact that $\varphi_1(0) = \varphi_2(0)$, because the domain is not rectangular and cannot be transformed into a regular domain without the appearance of some degenerate terms in the equation.

We look for some conditions on the functions $c(t)$, $\varphi_1(t)$ and $\varphi_2(t)$ such that (3.1) admits a unique solution u belonging to the anisotropic Sobolev space $H^{1,2}(\Omega)$.

We assume that there exist positive constants c_1 and c_2, such that

$$c_1 \leq c(t) \leq c_2, \quad \text{for all } t \in (0,T), \tag{3.3}$$

and we note that

$$\|u\|_{L^2(I_t)} = \Big(\int_{\varphi_1(t)}^{\varphi_2(t)} |u(t,x)|^2 \,\mathrm{d}x\Big)^{1/2},$$

$$\|u\|_{L^\infty(I_t)} = \sup_{x \in I_t} |u(t,x)|.$$

To establish the existence of a solution to (3.1), we also assume that

$$|\varphi'(t)| \leq \gamma \quad \text{for all } t \in [0,T], \tag{3.4}$$

where γ is a positive constant and $\varphi(t) = \varphi_2(t) - \varphi_1(t)$ for all $t \in [0,T]$.

Remark 3.1. *Once problem* (3.1) *is solved, we can deduce the solution of the problem*

$$\begin{aligned} \partial_t u(t,x) + a(t)u(t,x)\partial_x u(t,x) - b(t)\partial_x^2 u(t,x) = f(t,x) \quad (t,x) \in \Omega, \\ u(t,\varphi_1(t)) = u(t,\varphi_2(t)) = 0 \quad t \in (0,T). \end{aligned} \tag{3.5}$$

Indeed, consider the case where $a(t)$ *and* $b(t)$ *are positive and bounded functions for all* $t \in [0,T]$.

Let h *be defined by* $h : [0,T] \to [0,T']$

$$h(t) = \int_0^t b(s)\mathrm{d}s,$$

we put $\psi_i = \varphi_i \circ h^{-1}$ *where* $i = 1,2$. *Using the change of variables*

$$t' = h(t), \quad v(t',x) = u(t,x) \tag{3.6}$$

(3.5) *becomes equivalent to* (3.1), *because it may be written as follows*

$$\begin{aligned} \partial_{t'} v(t',x) + c(t')v(t',x)\partial_x v(t',x) - \partial_x^2 v(t',x) = g(t',x) \quad (t',x) \in \Omega', \\ v(t',\psi_1(t')) = v(t',\psi_2(t')) = 0, \quad t' \in (0,T'), \end{aligned}$$

where $c(t') = \dfrac{a(t)}{b(t)}$, $g(t',x) = \dfrac{f(t,x)}{b(t)}$, $\Omega' = \{(t',x) \in \mathbb{R}^2;\ 0 < t' < T',\ x \in I_{t'}\}$ *and* $T' = \int_0^T b(s)\mathrm{d}s$.

For the study of problem (3.1) we will follow the method used in [42], which consists in observing that this problem admits a unique solution in domains that can be transformed into rectangles, i.e., when $\varphi_1(0) \neq \varphi_2(0)$.

When φ_1 and φ_2 are monotone near 0, we solve in Section 3.2 the problem in a triangular domain: We approximate this domain by a sequence of subdomains $(\Omega_n)_{n\in\mathbb{N}}$. Then we establish an a priori estimate of the type

$$\|u_n\|^2_{H^{1,2}(\Omega_n)} \le K\|f_n\|^2_{L^2(\Omega_n)} \le K\|f\|^2_{L^2(\Omega)},$$

where u_n is the solution of (3.1) in Ω_n and K is a constant independent of n. This inequality allows us to pass to the limit in n. Section 3.3 is devoted to problem (3.1) in the case when φ_1 and φ_2 are monotone on $(0, T)$.

The main result of this chapter is as follows:

Theorem 3.2. *Assume that c and $(\varphi_i(t))_{i=1,2}$ satisfy the conditions* (3.2), (3.3) *and* (3.4). *Then, the problem*

$$\partial_t u(t,x) + c(t)u(t,x)\partial_x u(t,x) - \partial_x^2 u(t,x) = f(t,x) \quad (t,x) \in \Omega,$$
$$u(t,\varphi_1(t)) = u(t,\varphi_2(t)) = 0 \quad t \in (0,T),$$

admits in the triangular domain Ω a unique solution $u \in H^{1,2}(\Omega)$ in the following cases:

Case 1. *φ_1 (resp φ_2) is a decreasing (resp increasing) function on $(0, T)$.*

Case 2. *φ_1 (resp φ_2) is a decreasing (resp increasing) function only near* 0.

Theses cases will be proved in Section 3.2 and Section 3.3, respectively.

3.2 Proof of Theorem 3.2, Case 1

Let

$$\Omega = \left\{(t,x) \in \mathbb{R}^2 : 0 < t < T,\ x \in I_t\right\},$$
$$I_t = \{x \in \mathbb{R} : \varphi_1(t) < x < \varphi_2(t),\ t \in (0,T)\},$$

with $\varphi_1(0) = \varphi_2(0)$ and $\varphi_1(T) < \varphi_2(T)$.

For each $n \in \mathbb{N}^\star$, we define

$$\Omega_n = \left\{(t,x) \in \mathbb{R}^2 : \frac{1}{n} < t < T,\ x \in I_t\right\},$$

and we set $f_n = f_{|\Omega_n}$, where f is given in $L^2(\Omega)$. By Theorem 2.9 there exists a solution $u_n \in H^{1,2}(\Omega_n)$ of the problem

$$\begin{cases} \partial_t u_n(t,x) + c(t)u_n(t,x)\partial_x u_n(t,x) - \partial_x^2 u_n(t,x) = f_n(t,x) \quad (t,x) \in \Omega_n, \\ u_n(\frac{1}{n}, x) = 0, \quad \varphi_1(\frac{1}{n}) < x < \varphi_2(\frac{1}{n}), \\ u_n(t,\varphi_1(t)) = u_n(t,\varphi_2(t)) = 0 \quad t \in [\frac{1}{n}, T], \end{cases} \tag{3.7}$$

in Ω_n.

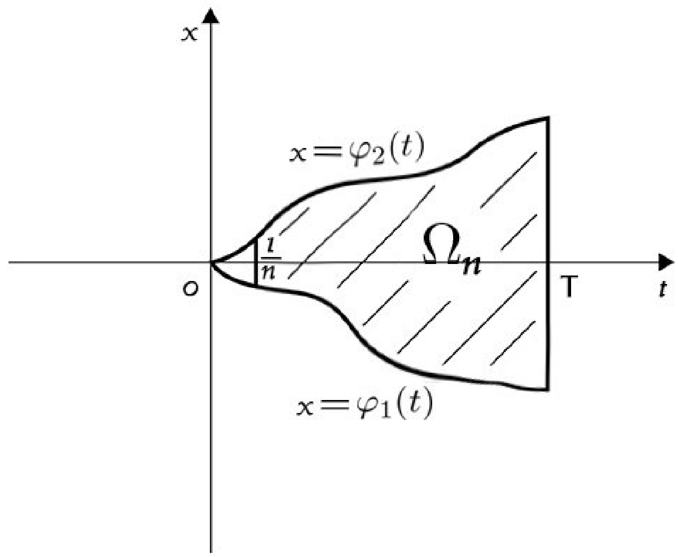

Figure 3.1: A non parabolic domain

To prove Case 1 of Theorem 3.2, we have to pass to the limit in (3.7). For this purpose we need the following result.

Proposition 3.3. *Under the assumptions of Theorem 3.2, there exists a positive constant K independent of n such that*

$$\|u_n\|^2_{H^{1,2}(\Omega_n)} \leq K\|f_n\|^2_{L^2(\Omega_n)} \leq K\|f\|^2_{L^2(\Omega)}.$$

To prove this proposition we need some preliminary results.

Lemma 3.4. *There exists a positive constant K_1 independent of n such that*

$$\|u_n\|^2_{L^2(\Omega_n)} \leq K_1\|\partial_x u_n\|^2_{L^2(\Omega_n)}, \tag{3.8}$$

$$\|\partial_x u_n\|^2_{L^2(\Omega_n)} \leq K_1\|f_n\|^2_{L^2(\Omega_n)}. \tag{3.9}$$

Proof. We have

$$|u_n|^2 = \Big| \int_{\varphi_1(t)}^{x} \partial_s u_n \,\mathrm{d}s \Big|^2 \leq (x - \varphi_1(t)) \int_{\varphi_1(t)}^{x} |\partial_s u_n|^2 \,\mathrm{d}s.$$

Integrating from $\varphi_1(t)$ to $\varphi_2(t)$, we obtain

$$\int_{\varphi_1(t)}^{\varphi_2(t)} |u_n|^2 \,\mathrm{d}x \leq \int_{\varphi_1(t)}^{\varphi_2(t)} \left((x - \varphi_1(t)) \int_{\varphi_1(t)}^{x} |\partial_s u_n|^2 \,\mathrm{d}s \right) \mathrm{d}x,$$

hence

$$\int_{\varphi_1(t)}^{\varphi_2(t)} |u_n|^2 \,\mathrm{d}x \leq (\varphi_2(t) - \varphi_1(t)) \int_{\varphi_1(t)}^{\varphi_2(t)} \int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \,\mathrm{d}x \,\mathrm{d}x,$$

and

$$\int_{\varphi_1(t)}^{\varphi_2(t)} |u_n|^2 \,\mathrm{d}x \leq (\varphi_2(t) - \varphi_1(t))^2 \int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \,\mathrm{d}x.$$

Then, there exists a positive constant K_1 independent of n such that

$$\|u_n\|^2_{L^2(I_t)} \leq K_1 \|\partial_x u_n\|^2_{L^2(I_t)},$$

integrating between $\frac{1}{n}$ and T we obtain inequality (3.8).

Now, multiplying both sides of (3.7) by u_n and integrating between $\varphi_1(t)$ and $\varphi_2(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt} \int_{\varphi_1(t)}^{\varphi_2(t)} u_n^2 \,\mathrm{d}x + c(t) \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u_n u_n^2 \,\mathrm{d}x - \int_{\varphi_1(t)}^{\varphi_2(t)} u_n \partial_x^2 u_n \,\mathrm{d}x = \int_{\varphi_1(t)}^{\varphi_2(t)} f_n u_n \,\mathrm{d}x.$$

Integration by parts gives

$$c(t) \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u_n u_n^2 \,\mathrm{d}x = \frac{c(t)}{3} \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x (u_n)^3 \,\mathrm{d}x = 0;$$

then

$$\frac{1}{2}\frac{d}{dt} \int_{\varphi_1(t)}^{\varphi_2(t)} u_n^2 \,\mathrm{d}x + \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^2 \,\mathrm{d}x = \int_{\varphi_1(t)}^{\varphi_2(t)} f_n u_n \,\mathrm{d}x. \tag{3.10}$$

By integrating (3.10) from $\frac{1}{n}$ to T, we find that

$$\frac{1}{2}\|u_n(T,x)\|^2_{L^2(I_T)} + \int_{1/n}^{T} \|\partial_x u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s$$

$$\leq \int_{1/n}^{T} \|f_n(s)\|_{L^2(I_t)}\|u_n(s)\|_{L^2(I_t)}\,\mathrm{d}s.$$

Using the elementary inequality

$$|rs| \leq \frac{\varepsilon}{2}r^2 + \frac{s^2}{2\varepsilon}, \quad \forall r,s \in \mathbb{R},\ \forall \varepsilon > 0, \tag{3.11}$$

with $\varepsilon = K_1$, we obtain

$$\frac{1}{2}\|u_n(T,x)\|^2_{L^2(I_T)} + \int_{1/n}^{T} \|\partial_x u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s$$

$$\leq \frac{K_1}{2}\int_{1/n}^{T} \|f_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s + \frac{1}{2K_1}\int_{1/n}^{T} \|u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s.$$

Thanks to (3.8), we have

$$\frac{1}{2K_1}\int_{1/n}^{T} \|u_n\|^2_{L^2(I_t)}\,\mathrm{d}s \leq \frac{1}{2}\int_{1/n}^{T} \|\partial_x u_n\|^2_{L^2(I_t)}\,\mathrm{d}s,$$

therefore

$$\|u_n(T,x)\|^2_{L^2(I_T)} + \int_{1/n}^{T} \|\partial_x u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s \leq K_1\int_{1/n}^{T} \|f_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s,$$

consequently

$$\|\partial_x u_n\|^2_{L^2(\Omega_n)} \leq K_1\|f_n\|^2_{L^2(\Omega_n)}.$$

□

Corollary 3.5. *There exists a positive constant K_2 independent of n, such that for all $t \in [\frac{1}{n}, T]$,*

$$\|\partial_x u_n\|^2_{L^2(I_t)} + \int_{1/n}^{T} \|\partial_x^2 u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s \leq K_2.$$

Proof. Multiplying both sides of (3.7) by $\partial_x^2 u_n$ and integrating between $\varphi_1(t)$ and $\varphi_2(t)$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^2 \mathrm{d}x + \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x^2 u_n)^2 \,\mathrm{d}x \\ &= -\int_{\varphi_1(t)}^{\varphi_2(t)} f_n \partial_x^2 u_n \,\mathrm{d}x + c(t)\int_{\varphi_1(t)}^{\varphi_2(t)} u_n \partial_x u_n \partial_x^2 u_n \,\mathrm{d}x. \end{aligned} \tag{3.12}$$

Using Cauchy-Schwarz inequality, (3.11) with $\varepsilon = \frac{1}{2}$ leads to

$$\begin{aligned} \left|\int_{\varphi_1(t)}^{\varphi_2(t)} f_n \partial_x^2 u_n \,\mathrm{d}x\right| &\leq \left(\int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x^2 u_n|^2 \,\mathrm{d}x\right)^{1/2} \left(\int_{\varphi_1(t)}^{\varphi_2(t)} |f_n|^2 \,\mathrm{d}x\right)^{1/2} \\ &\leq \frac{1}{4}\int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x^2 u_n|^2 \,\mathrm{d}x + \int_{\varphi_1(t)}^{\varphi_2(t)} |f_n|^2 \,\mathrm{d}x. \end{aligned} \tag{3.13}$$

Now, we have to estimate the last term of (3.12). An integration by parts gives

$$\begin{aligned} \int_{\varphi_1(t)}^{\varphi_2(t)} u_n \partial_x u_n \partial_x^2 u_n \,\mathrm{d}x &= \int_{\varphi_1(t)}^{\varphi_2(t)} u_n \partial_x \left(\frac{1}{2}(\partial_x u_n)^2\right) \mathrm{d}x \\ &= -\frac{1}{2}\int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^3 \,\mathrm{d}x. \end{aligned}$$

Since $\partial_x u_n$ satisfies $\int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u_n \,\mathrm{d}x = 0$ we deduce that the continuous function $\partial_x u_n$ is zero at some point $\xi(t) \in (\varphi_1(t), \varphi_2(t))$, and by integrating $2\partial_x u_n \partial_x^2 u_n$ between $\xi(t)$ and x, we obtain

$$2\int_{\xi(t)}^{x} \partial_s u_n \partial_s^2 u_n \,\mathrm{d}s = \int_{\xi(t)}^{x} \partial_s (\partial_s u_n)^2 \,\mathrm{d}s = (\partial_x u_n)^2,$$

the Cauchy-Schwartz inequality gives

$$\|\partial_x u_n\|_{L^\infty(I_t)}^2 \leq 2\|\partial_x u_n\|_{L^2(I_t)} \|\partial_x^2 u_n\|_{L^2(I_t)},$$

but

$$\|\partial_x u_n\|^3_{L^3(I_t)} \leq \|\partial_x u_n\|^2_{L^2(I_t)} \|\partial_x u_n\|_{L^\infty(I_t)},$$

so, (3.1) yields

$$\left| \int_{\varphi_1(t)}^{\varphi_2(t)} c(t) u_n \partial_x u_n \partial_x^2 u_n \, \mathrm{d}x \right| \leq \left(\int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x^2 u_n|^2 \, \mathrm{d}x \right)^{1/4} \left(c_2^{4/5} \int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \, \mathrm{d}x \right)^{5/4}.$$

Finally, by Young's inequality $|AB| \leq \frac{|A|^p}{p} + \frac{|B|^{p'}}{p'}$, with $1 < p < \infty$ and $p' = \frac{p}{p-1}$, choosing $p = 4$ (then $p' = \frac{4}{3}$)

$$A = \left(\int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x^2 u_n|^2 \, \mathrm{d}x \right)^{1/4}$$

and

$$B = \left(c_2^{4/5} \int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \, \mathrm{d}x \right)^{5/4},$$

the estimate of the last term of (3.12) becomes

$$\begin{aligned} &\left| \int_{\varphi_1(t)}^{\varphi_2(t)} c(t) u_n \partial_x u_n \partial_x^2 u_n \, \mathrm{d}x \right| \\ &\leq \frac{1}{4} \int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x^2 u_n|^2 \, \mathrm{d}x + \frac{3}{4} c_2^{4/3} \left(\int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \, \mathrm{d}x \right)^{5/3}. \end{aligned} \tag{3.14}$$

Let us return to (3.12): By integrating between $\frac{1}{n}$ and t, from the estimates (3.13) and

(3.14), we obtain

$$\begin{aligned}
&\frac{1}{2}\|\partial_x u_n\|^2_{L^2(I_t)} + \int_{1/n}^{t} \|\partial_x^2 u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s \\
&\leq +\frac{1}{4}\int_{1/n}^{t} \|\partial_x^2 u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s + \int_{1/n}^{t} \|f_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s \\
&\quad + \frac{1}{4}\int_{1/n}^{t} \|\partial_x^2 u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s + \frac{3}{4}c_2^{4/3}\int_{1/n}^{t} \|\partial_x u_n(s)\|^{10/3}_{L^2(I_t)}\,\mathrm{d}s,
\end{aligned}$$

then

$$\begin{aligned}
&\|\partial_x u_n\|^2_{L^2(I_t)} + \int_{1/n}^{t} \|\partial_x^2 u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s \\
&\leq 2\int_{1/n}^{t} \|f_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s + \frac{3}{2}c_2^{4/3}\int_{1/n}^{t} \left(\|\partial_x u_n(s)\|^2_{L^2(I_t)}\right)^{5/3}\mathrm{d}s.
\end{aligned}$$

$f_n \in L^2(\Omega_n)$, then there exists a constant c_3 such that

$$\begin{aligned}
&\|\partial_x u_n\|^2_{L^2(I_t)} + \int_{1/n}^{t} \|\partial_x^2 u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s \\
&\leq c_3 + \frac{3}{2}c_2^{4/3}\int_{1/n}^{t} \left(\|\partial_x u_n(s)\|^2_{L^2(I_t)}\right)^{2/3} \|\partial_x u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s.
\end{aligned}$$

Consequently, the function

$$\varphi(t) = \|\partial_x u_n\|^2_{L^2(I_t)} + \int_{1/n}^{t} \|\partial_x^2 u_n(s)\|^2_{L^2(I_t)}\,\mathrm{d}s$$

satisfies the inequality

$$\varphi(t) \leq c_3 + \int_{1/n}^{t} \left(\frac{3}{2}c_2^{4/3}\|\partial_x u_n(s)\|^{4/3}_{L^2(I_t)}\right)\varphi(s)\mathrm{d}s,$$

Gronwall's inequality shows that

$$\varphi(t) \leq c_3 \exp\left(\int_{1/n}^{t} \frac{3}{2}c_2^{4/3}\|\partial_x u_n(s)\|^{4/3}_{L^2(I_t)}\mathrm{d}s\right).$$

According to Lemma 3.4 the integral $\int_{1/n}^{t} \|\partial_x u_n\|_{L^2(I_t)}^{4/3} \mathrm{d}s$ is bounded by a constant independent of n. So there exists a positive constant K_2 such that

$$\|\partial_x u_n\|_{L^2(I_t)}^2 + \int_{1/n}^{T} \|\partial_x^2 u_n(s)\|_{L^2(I_t)}^2 \, \mathrm{d}s \leq K_2.$$

□

Lemma 3.6. *There exists a constant K_3 independent of n such that*

$$\|\partial_t u_n\|_{L^2(\Omega_n)}^2 + \|\partial_x^2 u_n\|_{L^2(\Omega_n)}^2 \leq K_3 \|f_n\|_{L^2(\Omega_n)}^2.$$

Then Theorem 3.3 is a direct consequence of Lemmas 3.4 and 3.6.

Proof. To prove Lemma 3.6, we develop the inner product in $L^2(\Omega_n)$,

$$\begin{aligned}\|f_n\|_{L^2(\Omega_n)}^2 &= (\partial_t u_n + c(t) u_n \partial_x u_n - \partial_x^2 u_n, \partial_t u_n + c(t) u_n \partial_x u_n - \partial_x^2 u_n)_{L^2(\Omega_n)} \\ &= \|\partial_t u_n\|_{L^2(\Omega_n)}^2 + \|\partial_x^2 u_n\|_{L^2(\Omega_n)}^2 + \|c(t) u_n \partial_x u_n\|_{L^2(\Omega_n)}^2 \\ &\quad - 2(\partial_t u_n, \partial_x^2 u_n)_{L^2(\Omega_n)} + 2(\partial_t u_n, c(t) u_n \partial_x u_n)_{L^2(\Omega_n)} \\ &\quad - 2(c(t) u_n \partial_x u_n, \partial_x^2 u_n)_{L^2(\Omega_n)},\end{aligned}$$

so,

$$\begin{aligned}&\|\partial_t u_n\|_{L^2(\Omega_n)}^2 + \|\partial_x^2 u_n\|_{L^2(\Omega_n)}^2 \\ &= \|f_n\|_{L^2(\Omega_n)}^2 - \|c(t) u_n \partial_x u_n\|_{L^2(\Omega_n)}^2 + 2(c(t) u_n \partial_x u_n, \partial_x^2 u_n)_{L^2(\Omega_n)} \\ &\quad - 2(\partial_t u_n, c(t) u_n \partial_x u_n)_{L^2(\Omega_n)} + 2(\partial_t u_n, \partial_x^2 u_n)_{L^2(\Omega_n)}.\end{aligned} \tag{3.15}$$

Using (3.3) and (3.11) with $\varepsilon = 1/2$, we obtain

$$\left|-2(\partial_t u_n, c(t) u_n \partial_x u_n)_{L^2(\Omega_n)}\right| \leq \frac{1}{2} \|\partial_t u_n\|_{L^2(\Omega_n)}^2 + 2c_2^2 \|u_n \partial_x u_n\|_{L^2(\Omega_n)}^2, \tag{3.16}$$

and

$$\left|2(c(t) u_n \partial_x u_n, \partial_x^2 u_n)_{L^2(\Omega_n)}\right| \leq 2c_2^2 \|u_n \partial_x u_n\|_{L^2(\Omega_n)}^2 + \frac{1}{2} \|\partial_x^2 u_n\|_{L^2(\Omega_n)}^2. \tag{3.17}$$

Now calculating the last term of (3.15),

$$
\begin{aligned}
(\partial_t u_n, \partial_x^2 u_n)_{L^2(\Omega_n)} &= -\int_{1/n}^{T}\int_{\varphi_1(t)}^{\varphi_2(t)} \partial_t(\partial_x u_n)\partial_x u_n \,\mathrm{d}x\mathrm{d}t + \int_{1/n}^{T} [\partial_t u_n \partial_x u_n]_{\varphi_1(t)}^{\varphi_2(t)} \,\mathrm{d}t \\
&= -\frac{1}{2}\int_{1/n}^{T}\int_{\varphi_1(t)}^{\varphi_2(t)} \partial_t(\partial_x u_n)^2 \,\mathrm{d}x\mathrm{d}t + \int_{1/n}^{T} [\partial_t u_n \partial_x u_n]_{\varphi_1(t)}^{\varphi_2(t)} \,\mathrm{d}t \\
&= -\frac{1}{2}\Big[\int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^2 \,\mathrm{d}x\Big]_{1/n}^{T} + \int_{1/n}^{T} [\partial_t u_n \partial_x u_n]_{\varphi_1(t)}^{\varphi_2(t)} \,\mathrm{d}t \\
&= -\frac{1}{2}\int_{\varphi_1(T)}^{\varphi_2(T)} (\partial_x u_n)^2(T,x) \,\mathrm{d}x + \frac{1}{2}\int_{\varphi_1(1/n)}^{\varphi_2(1/n)} (\partial_x u_n)^2(\frac{1}{n},x) \,\mathrm{d}x \\
&\quad + \int_{1/n}^{T} \partial_t u_n(t,\varphi_2(t))\partial_x u_n(t,\varphi_2(t)) \,\mathrm{d}t \\
&\quad - \int_{1/n}^{T} \partial_t u_n(t,\varphi_1(t))\partial_x u_n(t,\varphi_1(t)) \,\mathrm{d}t.
\end{aligned}
$$

According to the boundary conditions, we have

$$
\partial_t u_n(t,\varphi_i(t)) + \varphi_i'(t)\partial_x u_n(t,\varphi_i(t)) = 0, \quad i = 1,2,
$$

so

$$
\begin{aligned}
(\partial_t u_n, \partial_x^2 u_n)_{L^2(\Omega_n)} &= -\frac{1}{2}\int_{\varphi_1(T)}^{\varphi_2(T)} (\partial_x u_n)^2(T,x) \,\mathrm{d}x - \int_{1/n}^{T} \varphi_2'(t)(\partial_x u_n(t,\varphi_2(t)))^2 \,\mathrm{d}t \\
&\quad + \int_{1/n}^{T} \varphi_1'(t)(\partial_x u_n(t,\varphi_1(t)))^2 \,\mathrm{d}t,
\end{aligned}
$$

it follows that

$$
(\partial_t u_n, \partial_x^2 u_n) \leq 0. \tag{3.18}
$$

From (3.16), (3.17) and (3.18), (3.15) becomes

$$\begin{aligned}
&\|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial_x^2 u_n\|^2_{L^2(\Omega_n)} \\
&\leq \|f_n\|^2_{L^2(\Omega_n)} + \frac{1}{2}\|\partial_t u_n\|^2_{L^2(\Omega_n)} \\
&\quad + \frac{1}{2}\|\partial_x^2 u_n\|^2_{L^2(\Omega_n)} + 5c_2^2\|u_n\partial_x u_n\|^2_{L^2(\Omega_n)},
\end{aligned}$$

then

$$\|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial_x^2 u_n\|^2_{L^2(\Omega_n)} \leq 2\|f_n\|^2_{L^2(\Omega_n)} + 10c_2^2\|u_n\partial_x u_n\|^2_{L^2(\Omega_n)}. \tag{3.19}$$

On the other hand, using the injection of $H_0^1(I_t)$ in $L^\infty(I_t)$, we obtain

$$\begin{aligned}
\left|\int_{1/n}^{T}\int_{\varphi_1(t)}^{\varphi_2(t)} (u_n\partial_x u_n)^2\,\mathrm{d}x\,\mathrm{d}t\right| &\leq \int_{1/n}^{T}\left(\|u_n\|^2_{L^\infty(I_t)}\int_{\varphi_1(t)}^{\varphi_2(t)}|\partial_x u_n|^2\,\mathrm{d}x\right)\mathrm{d}t \\
&\leq \int_{1/n}^{T}\|u_n\|^2_{H_0^1(I_t)}\|\partial_x u_n\|^2_{L^2(I_t)}\,\mathrm{d}t \\
&\leq \|u_n\|^2_{L^\infty\left(\frac{1}{n},T;H_0^1(I_t)\right)}\|\partial_x u_n\|^2_{L^2(\Omega_n)},
\end{aligned}$$

According to Corollary 3.5, $\|u_n\|^2_{L^\infty\left(\frac{1}{n},T;H_0^1(I_t)\right)}$ is bounded, then by (3.9) and (3.19), there exists a constant K_3 independent of n, such that

$$\|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial_x^2 u_n\|^2_{L^2(\Omega_n)} \leq K_3\|f_n\|^2_{L^2(\Omega_n)}.$$

However,

$$\|f_n\|^2_{L^2(\Omega_n)} \leq \|f\|^2_{L^2(\Omega)},$$

then, from lemmas 3.4 and 3.6 , there exists a constant K independent of n, such that

$$\|u_n\|^2_{H^{1,2}(\Omega_n)} \leq K\|f_n\|^2_{L^2(\Omega_n)} \leq K\|f\|^2_{L^2(\Omega)}.$$

This completes the proof. □

Existence and uniqueness

Choose a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of the domains defined previously, such that $\Omega_n \subseteq \Omega$, as $n \longrightarrow +\infty$ then $\Omega_n \longrightarrow \Omega$.

Consider $u_n \in H^{1,2}(\Omega_n)$ the solution of

$$\begin{cases} \partial_t u_n(t,x) + c(t)u_n(t,x)\partial_x u_n(t,x) - \partial_x^2 u_n(t,x) = f_n(t,x) \quad (t,x) \in \Omega_n, \\ u_n(\frac{1}{n}, x) = 0 \quad \varphi_1(\frac{1}{n}) < x < \varphi_2(\frac{1}{n}), \\ u_n(t, \varphi_1(t)) = u_n(t, \varphi_2(t)) = 0 \quad t \in]\frac{1}{n}, T[. \end{cases}$$

We know that a solution u_n exists by the Theorem 2.9. Let $\widetilde{u_n}$ be the extension by zero of u_n outside Ω_n. From the proposition 3.3 results the inequality

$$\|\widetilde{u_n}\|^2_{L^2(\Omega_n)} + \|\partial_t \widetilde{u_n}\|^2_{L^2(\Omega_n)} + \|\partial_x \widetilde{u_n}\|^2_{L^2(\Omega_n)} + \|\partial_x^2 \widetilde{u_n}\|^2_{L^2(\Omega_n)} \le C\|f\|^2_{L^2(\Omega)}.$$

This implies that $\widetilde{u_n}$, $\partial_t \widetilde{u_n}$ and $\partial_x^j \widetilde{u_n}$, $j = 1, 2$ are bounded in $L^2(\Omega_n)$, from Corollary 3.5 $\widetilde{u_n}\partial_x \widetilde{u_n}$ is bounded in $L^2(\Omega_n)$. So, it is possible to extract a subsequence from u_n, still denoted u_n such that

$$\partial_t \widetilde{u_n} \to \partial_t u \quad \text{weakly in} \quad L^2(\Omega),$$

$$\partial_x^2 \widetilde{u_n} \to \partial_x^2 u \quad \text{weakly in} \quad L^2(\Omega),$$

$$\widetilde{u_n}\partial_x \widetilde{u_n} \to u\partial_x u \quad \text{weakly in } L^2(\Omega).$$

Then $u \in H^{1,2}(\Omega)$ is solution to problem (3.1).

For the uniqueness, let us observe that any solution $u \in H^{1,2}(\Omega)$ of problem (3.1) is in $L^\infty(0, T, H_0^1(I_t))$. Indeed, by the same way as in Corollary 3.5, we prove that there exists a positive constant K_2 such that for all $t \in [0, T]$

$$\|\partial_x u\|^2_{L^2(I_t)} + \int_0^T \|\partial_x^2 u(s)\|^2_{L^2(I_t)} \,\mathrm{d}s \le K_2.$$

Let $u_1, u_2 \in H^{1,2}(\Omega)$ be two solutions of (2.3). We put $u = u_1 - u_2$. It is clear that $u \in L^\infty(0, T, H_0^1(I_t))$. The equations satisfied by u_1 and u_2 leads to

$$\int_{\varphi_1(t)}^{\varphi_2(t)} [\partial_t u w + c(t)uw\partial_x u_1 + c(t)u_2 w\partial_x u + \partial_x u \partial_x w] \,\mathrm{d}x = 0.$$

Taking, for $t \in [0, T]$, $w = u$ as a test function, we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I_t)} + \|\partial_x u\|^2_{L^2(I_t)} \\ &= -c(t)\int_{\varphi_1(t)}^{\varphi_2(t)} u^2 \partial_x u_1 \,\mathrm{d}x - c(t)\int_{\varphi_1(t)}^{\varphi_2(t)} u_2 u \partial_x u \,\mathrm{d}x. \end{aligned} \tag{3.20}$$

An integration by parts gives

$$c(t)\int_{\varphi_1(t)}^{\varphi_2(t)} u^2\partial_x u_1 \,\mathrm{d}x = -2c(t)\int_{\varphi_1(t)}^{\varphi_2(t)} u\partial_x u u_1 \,\mathrm{d}x,$$

then (3.20) becomes

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I_t)} + \|\partial_x u\|^2_{L^2(I_t)} = \int_{\varphi_1(t)}^{\varphi_2(t)} c(t)(2u_1 - u_2)u\partial_x u \,\mathrm{d}x.$$

By (3.3) and inequality (3.11) with $\varepsilon = 2$, we obtain

$$\begin{aligned}&\left|\int_{\varphi_1(t)}^{\varphi_2(t)} c(t)(2u_1 - u_2)u\partial_x u \,\mathrm{d}x\right| \\ &\leq \frac{1}{4}c_2^2(2\|u_1\|_{L^\infty(0,T,H_0^1(I_t))} + \|u_2\|_{L^\infty(0,T,H_0^1(I_t))})^2\|u\|^2_{L^2(I_t)} + \|\partial_x u\|^2_{L^2(I_t)}.\end{aligned}$$

So, we deduce that there exists a non-negative constant D, such as

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I_t)} \leq D\|u\|^2_{L^2(I_t)},$$

and Gronwall's lemma leads to $u = 0$. This completes the proof of Theorem 3.2, Case 1.

3.3 Proof of Theorem 3.2, Case 2

In this case we set $\Omega = Q_1 \cup Q_2 \cup \Gamma_{T_1}$ where

$$\begin{aligned} Q_1 &= \left\{(t,x) \in \mathbb{R}^2 : 0 < t < T_1,\ x \in I_t\right\}, \\ Q_2 &= \left\{(t,x) \in \mathbb{R}^2 : T_1 < t < T,\ x \in I_t\right\}, \\ \Gamma_{T_1} &= \left\{(T_1,x) \in \mathbb{R}^2 : x \in I_{T_1}\right\}, \end{aligned}$$

with T_1 small enough. $f \in L^2(\Omega)$ and $f_i = f_{|Q_i}$, $i = 1, 2$.

Theorem 3.2, Case 1, applied to the domain Q_1, shows that there exists a unique solution $u_1 \in H^{1,2}(Q_1)$ of the problem

$$\begin{aligned}\partial_t u_1(t,x) + c(t)u_1(t,x)\partial_x u_1(t,x) - \partial_x^2 u_1(t,x)\\ = f_1(t,x) \quad (t,x) \in Q_1,\\ u_1(t,\varphi_1(t)) = u_1(t,\varphi_2(t)) = 0 \quad t \in (0,T_1).\end{aligned}$$

Lemma 3.7. *If $u \in H^{1,2}((T_1,T)\times(0,1))$, then $u_{|t=T_1} \in H^1(\{T_1\}\times(0,1))$.*

The above lemma is a special case of [34, Theorem 2.1, Vol. 2]. Using the transformation $[T_1,T]\times[0,1] \to Q_2$,

$$(t,x) \mapsto (t,y) = (t,(\varphi_2(t)-\varphi_1(t))x + \varphi_1(t))$$

we deduce from Lemma 3.7 the following result.

Lemma 3.8. *If $u \in H^{1,2}(Q_2)$, then $u_{|\Gamma_{T_1}} \in H^1(\Gamma_{T_1})$.*

We denote the trace $u_{1|\Gamma_{T_1}}$ by u_0 which is in the Sobolev space $H^1(\Gamma_{T_1})$ because $u_1 \in H^{1,2}(Q_1)$.
Theorem 2.9 applied to the domain Q_2, shows that there exists a unique solution $u_2 \in H^{1,2}(Q_2)$ of the problem

$$\begin{aligned}\partial_t u_2(t,x) + c(t)u_2(t,x)\partial_x u_2(t,x) - \partial_x^2 u_2(t,x) = f_2(t,x) \quad (t,x) \in Q_2,\\ u_2(T_1,x) = u_0(x) \quad \varphi_1(T_1) < x < \varphi_2(T_1),\\ u_2(t,\varphi_1(t)) = u_2(t,\varphi_2(t)) = 0 \quad t \in [T_1,T],\end{aligned}$$

putting

$$u = \begin{cases} u_1 & \text{in} \quad Q_1,\\ u_2 & \text{in} \quad Q_2,\end{cases}$$

we observe that $u \in H^{1,2}(\Omega)$ because $u_{1|\Gamma_{T_1}} = u_{2|\Gamma_{T_1}}$ and is a solution of the problem

$$\begin{aligned}\partial_t u(t,x) + c(t)u(t,x)\partial_x u(t,x) - \partial_x^2 u(t,x) = f(t,x) \quad (t,x) \in \Omega,\\ u(t,\varphi_1(t)) = u(t,\varphi_2(t)) = 0 \quad t \in (0,T).\end{aligned}$$

We prove the uniqueness of the solution by the same way as in Case 1.