Chapter 2

Existence of solutions to Burgers equation in domain that can be transformed into rectangle

In this chapter, we consider a non homogeneous Burgers problem with time variable coefficients subject to Cauchy-Dirichlet boundary conditions in a non rectangular domain. This domain will be transformed into a rectangle by a regular change of variables. The right-hand side of the equation is taken in L^2, and the initial condition is in the Sobolev space H_0^1. The goal is to establish the existence, the uniqueness and the regularity of the solution.

2.1 Existence of solutions to a parabolic problem with variable coefficients in a rectangle

In this section, we consider the semilinear parabolic problem

$$\begin{cases} \partial_t u(t,x) + p(t)u(t,x)\partial_x u(t,x) - q(t)\partial_x^2 u(t,x) + \\ \qquad\qquad r(t,x)\partial_x u(t,x) = f(t,x) \ (t,x) \in R, \\ u(0,x) = u_0(x) \quad x \in I, \\ u(t,a) = u(t,0) = 0 \quad t \in (0,T), \end{cases} \tag{2.1}$$

in the rectangle $R = (0,T) \times I$ where $I = (0,a)$, $a \in R^+$ (T is a positive finite number); $f \in L^2(R)$ and $u_0 \in H_0^1(I)$ are given functions.

We assume that the functions p, q depend only on t and the function r depends on t and x. We also suppose that there exist two positive constants α and β, such that

$$\begin{aligned} &\alpha \le p(t) \le \beta, \qquad \alpha \le q(t) \le \beta, \qquad \forall t \in [0,T] \\ \textit{and} \qquad &|\partial_x r(t,x)| \le \beta \quad \text{ou} \quad |r(t,x)| \le \beta \qquad \forall (t,x) \in R. \end{aligned} \tag{2.2}$$

In a paper by Morandi Cecchi *et al.* [37], the main result was the existence and uniqueness of a solution to the Burgers problem (with constant coefficients) in the anistropic Sobolev space

$$H^{1,2}(R) = \left\{ u \in L^2(R) : \partial_t u \in L^2(R),\ \partial_x u \in L^2(R),\ \partial_x^2 u \in L^2(R) \right\}$$

where R is a rectangle. The authors used a wrong inequality (namely $\int_\Omega M(u-M)^+(t)\,\mathrm{d}x \le M\|(u-M)^+(t)\|^2$) at the end of the proof of Theorem 2 (maximum principle); the inequality appears in the line 14, page 165 (and line 15 page 167). To rectify this part of the proof it suffices to show that $u \in L^\infty(Q)$. The proof given by the authors remains true only when $f = 0$ (but this was not the objective of their paper), this case being treated by Bressan in [9]. However, in our work, using another method, we prove a more general result concerning the existence, uniqueness and regularity of a solution to the Burgers problem with variable coefficients in a rectangle. Then, the existence, uniqueness and regularity of a solution to the Burgers problem in a domain that can be transformed into a rectangle.

The main result of This section is as follows:

Theorem 2.1. *If* $u_0 \in H_0^1(I)$, $f \in L^2(R)$ *and* p, q, r *satisfy the assumption* (2.2), *then Problem* (2.1) *admits a unique solution* $u \in H^{1,2}(R)$.

2.1.1 Resolution of the parabolic problem (2.1)

The proof of Theorem 2.1 is based on the Faedo-Galerkin method. We introduce approximate solution by reduction to the finite dimension. By the Faedo-Galerkin method, we obtain the existence of an approximate solution using an existence theorem of solutions for a system of ordinary differential equations. We approximate the equation of Problem (2.1) by a simple equation. Then we make the passage to the limit using a compactness argument.

Multiplying the equation of Problem (2.1) by a test function $w \in H_0^1(I)$, and integrating by parts from 0 to a, we obtain

$$\begin{aligned} \int_0^a \partial_t u w \,\mathrm{d}x + q(t)\int_0^a \partial_x u \partial_x w \,\mathrm{d}x + p(t)\int_0^a u\partial_x u w \,\mathrm{d}x + \int_0^a r(t,x)\partial_x u w \,\mathrm{d}x \\ = \int_0^a f w \,\mathrm{d}x,\ \forall w \in H_0^1(I), \quad t \in (0,T). \end{aligned} \tag{2.3}$$

This is the weak formulation of Problem (2.1). The solution of (2.3) satisfying the conditions of Problem (2.1) is called *weak solution*.

To prove the existence of a weak solution to (2.1), we choose the basis $(e_j)_{j\in\mathbb{N}^\star}$ of $L^2(I)$ defined as a subset of the eigenfunctions of $-\partial_x^2$ for the Dirichlet problem

$$\begin{aligned} -\partial_x^2 e_j &= \lambda_j e_j, \quad j \in \mathbb{N}^*, \\ e_j &= 0 \quad \text{on } \Gamma = \{0, a\}. \end{aligned}$$

More precisely,

$$e_j(x) = \frac{\sqrt{2}}{\sqrt{a}} \sin\frac{j\pi x}{a}, \quad \lambda_j = (\frac{j\pi}{a})^2, \quad \text{for } j \in \mathbb{N}^*.$$

As the family $(e_j)_{j\in\mathbb{N}^\star}$ is an orthonormal basis of $L^2(I)$, then it is an orthogonal basis of $H_0^1(I)$. In particular, for $v \in L^2(R)$, we can write

$$v = \sum_{k=1}^{\infty} b_k(t)e_k,$$

where $b_k = (v, e_k)_{L^2(I)}$ and the series converges in $L^2(I)$. Then, we introduce the approximate solution u_n by

$$u_n(t) = \sum_{j=1}^{n} c_j(t)e_j,$$
$$u_n(0) = u_{0n} = \sum_{j=1}^{n} c_j(0)e_j,$$

which has to satisfy the approximate problem

$$\begin{cases} \int\limits_0^a \left(\partial_t u_n + p(t)u_n\partial_x u_n\right) e_j \,\mathrm{d}x + q(t)\int\limits_0^a \partial_x u_n \partial_x e_j \,\mathrm{d}x \\ \qquad\qquad + \int\limits_0^a r(t,x)\partial_x u_n e_j \,\mathrm{d}x = \int\limits_0^a f e_j \,\mathrm{d}x, \\ u_n(0) = u_{0n}. \end{cases} \tag{2.4}$$

for all $j = 1, \dots, n$, and $0 \le t \le T$.

Remark 2.2. The coefficients $c_j(0)$ (which depend on j and n) will be chosen such that the sequence (u_{0n}) converges in $H_0^1(I)$ to u_0.

2.1.2 Solution of the approximate problem

Lemma 2.3. *For all j, there exists a unique solution u_n of Problem* (2.4).

Proof. As $e_1, \cdots, e_n$ are orthonormal in $L^2(I)$, then

$$\begin{aligned} \int\limits_0^a \partial_t u_n e_j \,\mathrm{d}x &= \sum_{i=1}^{n} c_i'(t) \int\limits_0^a e_i e_j \,\mathrm{d}x \\ &= c_j'(t). \end{aligned}$$

On the other hand,

$$-\partial_x^2 e_i = \lambda_i e_i,$$

then

$$\partial_x^2 u_n(t) = -\sum_{i=1}^{n} c_i(t)\lambda_i e_i.$$

Therefore, for all $t \in [0, T]$

$$\begin{aligned} -q(t)\int_0^a \partial_x^2 u_n e_j \,\mathrm{d}x &= q(t)\sum_{i=1}^{n} c_i(t)\lambda_i \int_0^a e_i e_j \,\mathrm{d}x \\ &= q(t)\lambda_j c_j(t). \end{aligned}$$

Now, if we introduce

$$f_j(t) = \int_0^a f e_j \,\mathrm{d}x,$$

$$k_j(t) = -p(t)\int_0^a u_n \partial_x u_n e_j \,\mathrm{d}x,$$

and

$$h_j(t) = -\int_0^a r(t,x)\partial_x u_n e_j \,\mathrm{d}x,$$

for $j \in \{1, \ldots, n\}$, then (2.4) is equivalent to the following system of n uncoupled linear ordinary differential equations:

$$c_j'(t) = -q(t)\lambda_j c_j(t) + k_j(t) + h_j(t) + f_j(t), \qquad j = 1, ..., n. \tag{2.5}$$

The terms $k_j(t), h_j(t)$ are well defined (because e_j and $r(t,x)$ are regular) and f_j is integrable (because $f \in L^2(R)$). Taking into account the initial condition $c_j(0)$, for each fixed $j \in \{1, \ldots, n\}$, (2.5) has a unique regular solution c_j in some interval $(0, T')$ with $T' \leq T$. In fact, we can prove here that $T' = T$. □

2.1.3 A priori estimate

Lemma 2.4. *There exists a positive constant K_1 independent of n, such that for all $t \in [0,T]$*

$$\|u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)} \,\mathrm{d}s \le K_1.$$

Proof. Multiplying (2.4) by c_j and summing for $j = 1, \dots, n$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^a u_n^2 \,\mathrm{d}x + q(t)\int_0^a (\partial_x u_n)^2 \,\mathrm{d}x - \frac{1}{2}\int_0^a \partial_x r(t,x) u_n^2 \,\mathrm{d}x = \int_0^a f u_n \,\mathrm{d}x.$$

Indeed, because of the boundary conditions, we have

$$p(t)\int_0^a u_n^2 \partial_x u_n \,\mathrm{d}x = \frac{p(t)}{3}\int_0^a \partial_x (u_n)^3 \,\mathrm{d}x = 0,$$

and an integration by parts gives

$$-\frac{1}{2}\int_0^a \partial_x r(t,x) u_n^2 \,\mathrm{d}x = \int_0^a r(t,x) u_n \partial_x u_n \,\mathrm{d}x.$$

Then, by integrating with respect to t ($t \in (0,T)$), and according to (2.2), we find that

$$\begin{aligned} &\frac{1}{2}\|u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)} \,\mathrm{d}s \\ &\le \frac{1}{2}\|u_{0n}\|^2_{L^2(I)} + \int_0^t \|f(s)\|_{L^2(I)} \|u_n(s)\|_{L^2(I)} \,\mathrm{d}s + \frac{\beta}{2}\int_0^t \|u_n(s)\|^2_{L^2(I)} \,\mathrm{d}s. \end{aligned}$$

By the elementary inequality

$$|rs| \le \frac{\varepsilon}{2} r^2 + \frac{s^2}{2\varepsilon}, \quad \forall r, s \in \mathbb{R},\ \forall \varepsilon > 0, \tag{2.6}$$

with $\varepsilon = \dfrac{2\alpha}{a^2}$, we obtain

$$\begin{aligned}&\frac{1}{2}\|u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s\\ &\leq \frac{1}{2}\|u_{0n}\|^2_{L^2(I)} + \frac{a^2}{4\alpha}\int_0^t \|f(s)\|^2_{L^2(I)}\,\mathrm{d}s\\ &+ \frac{\alpha}{a^2}\int_0^t \|u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s + \frac{\beta}{2}\int_0^t \|u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s,\end{aligned}$$

using Poincaré's inequality

$$\|u_n\|^2_{L^2(I)} \leq \frac{a^2}{2}\|\partial_x u_n\|^2_{L^2(I)},$$

then

$$\frac{\alpha}{a^2}\|u_n\|^2_{L^2(I)} \leq \frac{\alpha}{2}\|\partial_x u_n\|^2_{L^2(I)},$$

and we obtain,

$$\begin{aligned}&\|u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s\\ &\leq \|u_{0n}\|^2_{L^2(I)} + \frac{a^2}{2\alpha}\int_0^t \|f(s)\|^2_{L^2(I)}\,\mathrm{d}s + \beta\int_0^t \|u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}&\|u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s\\ &\leq \|u_{0n}\|^2_{L^2(I)} + \frac{a^2}{2\alpha}\int_0^t \|f(s)\|^2_{L^2(I)}\,\mathrm{d}s\\ &+ \beta\int_0^t \left(\|u_n(s)\|^2_{L^2(I)} + \alpha\int_0^s \|\partial_x u_n(\tau)\|^2_{L^2(I)}\,\mathrm{d}\tau\right)\mathrm{d}s.\end{aligned}$$

As the sequence (u_{0n}) converges in $H^1_0(I)$ to u_0 (see Remark 2.2) and $f \in L^2(R)$, there exists a positive constant C_1 independent of n such that

$$\|u_{0n}\|^2_{L^2(I)} + \frac{a^2}{2\alpha}\|f\|^2_{L^2(R)} \leq C_1$$

and

$$\|u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s$$
$$\leq C_1 + \beta \int_0^t \left(\|u_n(s)\|^2_{L^2(I)} + \alpha \int_0^s \|\partial_x u_n(\tau)\|^2_{L^2(I)}\,\mathrm{d}\tau \right) \mathrm{d}s,$$

then by Gronwall's inequality (see Corollary 1.33),

$$\|u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s \leq C_1 \exp(\beta t).$$

Taking $K_1 = C_1 \exp(\beta T)$, we obtain

$$\|u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x v_n(s)\|^2_{L^2(I)}\,\mathrm{d}s \leq K_1.$$

□

Lemma 2.5. *There exists a positive constant K_2 independent of n, such that for all $t \in [0,T]$*

$$\|\partial_x u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x^2 u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s \leq K_2.$$

Proof. As $-\partial_x^2 e_j = \lambda_j e_j$, we deduce that

$$\sum_{j=1}^n c_j(t)\lambda_j e_j = -\sum_{j=1}^n c_j(t)\partial_x^2 e_j = -\partial_x^2 u_n(t),$$

then, multiplying both sides of (2.4) by $c_j\lambda_j$ and summing for $j = 1, \ldots, n$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^a (\partial_x u_n)^2 \mathrm{d}x + q(t)\int_0^a (\partial_x^2 u_n)^2\,\mathrm{d}x \\ &= -\int_0^a f\partial_x^2 u_n\,\mathrm{d}x + \int_0^a r(t,x)\partial_x u_n \partial_x^2 u_n\,\mathrm{d}x + p(t)\int_0^a u_n\partial_x u_n \partial_x^2 u_n\,\mathrm{d}x.\end{aligned} \tag{2.7}$$

Using Cauchy-Schwartz inequality, (2.6) with $\varepsilon = \alpha/2$ leads to

$$\begin{aligned}\left|\int_0^a f\partial_x^2 u_n \,\mathrm{d}x\right| &\leq \left(\int_0^a |\partial_x^2 u_n|^2 \,\mathrm{d}x\right)^{1/2} \left(\int_0^a |f|^2 \,\mathrm{d}x\right)^{1/2} \\ &\leq \frac{\alpha}{4}\int_0^a |\partial_x^2 u_n|^2 \,\mathrm{d}x + \frac{1}{\alpha}\int_0^a |f|^2 \,\mathrm{d}x.\end{aligned} \tag{2.8}$$

By (2.2) and inequality (2.6) with $\varepsilon = \dfrac{\alpha}{2}$, it follows that

$$\left|\int_0^a r(t,x)\partial_x u_n \partial_x^2 u_n \,\mathrm{d}x\right| \leq \frac{\beta^2}{\alpha}\int_0^a |\partial_x u_n|^2 \,\mathrm{d}x + \frac{\alpha}{4}\int_0^a |\partial_x^2 u_n(s)| \,\mathrm{d}x. \tag{2.9}$$

Now, we have to estimate the last term of (2.7). An integration by parts gives

$$\begin{aligned}\int_0^a u_n \partial_x u_n \partial_x^2 u_n \,\mathrm{d}x &= \int_0^a u_n \partial_x \left(\frac{1}{2}(\partial_x u_n)^2\right) \mathrm{d}x \\ &= -\frac{1}{2}\int_0^a (\partial_x u_n)^3 \,\mathrm{d}x.\end{aligned}$$

Since $\partial_x u_n$ satisfies $\int_0^a \partial_x u_n \,\mathrm{d}x = 0$ we deduce that the continuous function $\partial_x u_n$ is zero at some point $y_{0n} \in (0,a)$, and by integrating $2\partial_x u_n \partial_x^2 u_n$ between y_{0n} and x, we obtain

$$|\partial_x u_n|^2 = \left|\int_{y_{0n}}^x \partial_x(\partial_x u_n)^2 \,\mathrm{d}x\right| = 2\left|\int_{y_{0n}}^x \partial_x u_n \partial_x^2 u_n \,\mathrm{d}x\right|,$$

the Cauchy-Schwarz inequality gives

$$\|\partial_x u_n\|^2_{L^\infty(I)} \leq 2\|\partial_x u_n\|_{L^2(I)}\|\partial_x^2 u_n\|_{L^2(I)}.$$

But

$$\|\partial_x u_n\|^3_{L^3(I)} \leq \|\partial_x u_n\|^2_{L^2(I)}\|\partial_x u_n\|_{L^\infty(I)}.$$

So, (2.2) yields

$$\left|\int_0^a p(t) u_n \partial_x u_n \partial_x^2 u_n \,\mathrm{d}x\right| \leq \left(\int_0^a |\partial_x^2 u_n|^2 \,\mathrm{d}x\right)^{1/4} \left(\beta^{4/5}\int_0^a |\partial_x u_n|^2 \,\mathrm{d}x\right)^{5/4}.$$

Finally, thanks to Young's inequality $|AB| \le \dfrac{|A|^p}{p} + \dfrac{|B|^{p'}}{p'}$, with $1 < p < \infty$ and $p' = \dfrac{p}{p-1}$, we have

$$\begin{aligned} |AB| &= \left| (\alpha^{1/p} A) \left(\alpha^{1/p'} \frac{B}{\alpha} \right) \right| \\ &\le \frac{\alpha}{p}|A|^p + \frac{\alpha}{p' \alpha^{p'}} |B|^{p'}. \end{aligned}$$

Choosing $p = 4$ (then $p' = \dfrac{4}{3}$) in the previous formula,

$$A = \left(\int_0^a |\partial_x^2 u_n|^2 \, \mathrm{d}x \right)^{1/4},$$

and

$$B = \left(\beta^{4/5} \int_0^a |\partial_x u_n|^2 \, \mathrm{d}x \right)^{5/4},$$

the estimate of the last term of (2.7) becomes

$$\left| \int_0^a p(t) u_n \partial_x u_n \partial_x^2 u_n \, \mathrm{d}x \right| \le \frac{\alpha}{4} \int_0^a |\partial_x^2 u_n|^2 \, \mathrm{d}x + \frac{3}{4} \frac{\beta^{4/3}}{\alpha^{1/3}} \left(\int_0^a |\partial_x u_n|^2 \, \mathrm{d}x \right)^{5/3}. \tag{2.10}$$

Let us return to inequality (2.7): By integrating between 0 and t, from the estimates (2.8), (2.9), and (2.10) we obtain

$$\begin{aligned} &\frac{1}{2} \|\partial_x u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x^2 u_n(s)\|^2_{L^2(I)} \, \mathrm{d}s \\ &\le \frac{1}{2} \|\partial_x u_{0n}\|^2_{L^2(I)} + \frac{\alpha}{4} \int_0^t \|\partial_x^2 u_n(s)\|^2_{L^2(I)} \, \mathrm{d}s + \frac{1}{\alpha} \int_0^t \|f(s)\|^2_{L^2(I)} \, \mathrm{d}s \\ &\quad + \frac{\beta^2}{\alpha} \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)} \, \mathrm{d}s + \frac{\alpha}{4} \int_0^t \|\partial_x^2 u_n(s)\|^2_{L^2(I)} \, \mathrm{d}s \\ &\quad + \frac{\alpha}{4} \int_0^t \|\partial_x^2 u_n(s)\|^2_{L^2(I)} \, \mathrm{d}s + \frac{3\beta^{4/3}}{4\alpha^{1/3}} \int_0^t \left(\|\partial_x u_n(s)\|^2_{L^2(I)} \right)^{5/3} \mathrm{d}s, \end{aligned}$$

multiplying both sides of the last inequality by 4, we obtain

$$\begin{aligned}
&2\|\partial_x u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x^2 u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s \\
&\leq 2\|\partial_x u_{0n}\|^2_{L^2(I)} + \frac{4}{\alpha}\int_0^t \|f(s)\|^2_{L^2(I)}\,\mathrm{d}s \\
&\quad + C_2 \int_0^t \left(\|\partial_x u_n(s)\|^2_{L^2(I)}\right)^{5/3} \mathrm{d}s + C_3 \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s,
\end{aligned}$$

where $C_2 = \dfrac{3\beta^{4/3}}{\alpha^{1/3}}$ and $C_3 = \dfrac{4\beta^2}{\alpha}$.

Observe that $f \in L^2(R))$, and $\|\partial_x u_{0n}\|^2_{L^2(I)}$ is bounded (see Remark 2.2). Then, there exists a constant C_4 such that

$$\begin{aligned}
&\|\partial_x u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x^2 u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s \\
&\leq C_4 + C_2 \int_0^t \left(\|\partial_x u_n(s)\|^2_{L^2(I)}\right)^{2/3} \|\partial_x u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s + C_3 \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s.
\end{aligned}$$

Consequently, the function

$$\varphi(t) = \|\partial_x u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x^2 u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s$$

satisfies the inequality

$$\varphi(t) \leq C_4 + \int_0^t (C_2\|\partial_x u_n(s)\|^{4/3}_{L^2(I)} + C_3)\varphi(s)\mathrm{d}s.$$

Gronwall's inequality shows that

$$\varphi(t) \leq C_4 \exp\left(\int_0^t (C_2\|\partial_x u_n(s)\|^{4/3}_{L^2(I)} + C_3)\mathrm{d}s\right).$$

According to Lemma 2.4 the integral $\int_0^t \|\partial_x u_n\|^{4/3}_{L^2(I)}\mathrm{d}s$ is bounded by a constant independent of n (and t).

So, there exists a positive constant K_2 such that

$$\|\partial_x u_n\|^2_{L^2(I)} + \alpha \int_0^t \|\partial_x^2 u_n(s)\|^2_{L^2(I)} \, \mathrm{d}s \leq K_2.$$

□

Lemma 2.6. *There exists a positive constant K_3 independent of n, such that for all $t \in [0, T]$*

$$\|\partial_t u_n\|^2_{L^2(R)} \leq K_3.$$

Proof. Let

$$g_n = f - p(t) u_n \partial_x u_n + q(t) \partial_x^2 u_n - r(t,x) \partial_x u_n.$$

To show that $\partial_t u_n$ is bounded in $L^2(R)$, we will first show that g_n is bounded in $L^2(R)$.

According to Lemmas 2.4 and 2.5, the terms $r(t,x)\partial_x u_n$ and $\beta(t)\partial_x^2 u_n$ are bounded in $L^2(R)$. On the other hand, by the hypothesis $f \in L^2(R)$. It remains only to show that $p(t) u_n \partial_x u_n \in L^2(R)$.

Lemma 2.4 proves that $\|u_n\|^2_{L^\infty(0,T;H_0^1(I))}$ is bounded. Then, using the injection of $H_0^1(I)$ in $L^\infty(I)$, we obtain

$$\begin{aligned}
\left| \int_0^T \int_0^a (p(t) u_n \partial_x u_n)^2 \, \mathrm{d}x \, \mathrm{d}t \right| &\leq \beta^2 \int_0^T \left(\|u_n\|^2_{L^\infty(I)} \int_0^a |\partial_x u_n|^2 \, \mathrm{d}x \right) \mathrm{d}t \\
&\leq \beta^2 C_I \int_0^T \|u_n\|^2_{H_0^1(I)} \|\partial_x u_n\|^2_{L^2(I)} \, \mathrm{d}t \\
&\leq \beta^2 C_I \|u_n\|^2_{L^\infty(0,T;H_0^1(I))} \|\partial_x u_n\|^2_{L^2(R)},
\end{aligned}$$

where C_I is a constant independent of n. Hence g_n is bounded in $L^2(R)$. So, $\partial_t u_n$ is also bounded in $L^2(R)$.

Indeed, from (2.4) for $j = 1, \ldots, n$, we have

$$\begin{aligned}
\int_0^a \partial_t u_n e_j \, \mathrm{d}x &= \int_0^a (f - p(t) u_n \partial_x u_n + q(t) \partial_x^2 u_n - r(t,x) \partial_x u_n) e_j \, \mathrm{d}x, \\
&= \int_0^a g_n e_j \, \mathrm{d}x,
\end{aligned}$$

multiplying both sides by c_j' and summing for $j = 1, \ldots, n$,

$$\|\partial_t u_n\|^2_{L^2(I)} = \int_0^a g_n \partial_t u_n \, \mathrm{d}x,$$

we deduce that $\|\partial_t u_n\|_{L^2(R)} \leq \|g_n\|_{L^2(R)}$. □

2.1.4 Existence and uniqueness

Lemmas 2.4, 2.5 and 2.6 show that the Galerkin approximation u_n is bounded in $L^\infty(0, T, L^2(I))$, and in $L^2(0, T, H^2(I))$, and $\partial_t u_n$ is bounded in $L^2(R)$. So, it is possible to extract a subsequence from u_n (that we continue to denote u_n) such that

$$u_n \to u \quad \text{weakly in } L^2(0, T, H^1_0(I)), \tag{2.11}$$

$$u_n \to u \quad \text{strongly in } L^2(0, T, L^2(I)) \text{ and a.e. in } R, \tag{2.12}$$

$$\partial_t u_n \to \partial_t u \quad \text{weakly in } L^2(R). \tag{2.13}$$

Lemma 2.7. *Under the assumptions of Theorem 2.1 , Problem* (2.1) *admits a weak solution* $u \in H^{1,2}(R)$.

Proof. Note that (2.13) implies

$$\int_0^T \int_0^a \partial_t u_n w \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_0^a \partial_t u w \, \mathrm{d}x \, \mathrm{d}t, \quad \forall w \in L^2(R).$$

From (2.11) and (2.12),

$$u_n \partial_x u_n \to u \partial_x u \quad \text{weakly in } \; L^2(R)\,,$$

then

$$\int_0^T \int_0^a p(t) u_n \partial_x u_n w \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_0^a p(t) u \partial_x u w \, \mathrm{d}x \, \mathrm{d}t, \quad \forall w \in L^2(R),$$

and

$$\int_0^T \int_0^a r(t,x) \partial_x u_n w \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_0^a r(t,x) \partial_x u w \, \mathrm{d}x \, \mathrm{d}t, \quad \forall w \in L^2(R).$$

Our goal is to use these properties to pass to the limit. In Problem (2.4), when $n \to +\infty$, for each fixed j, we have

$$\begin{aligned}&\int_0^a (\partial_t u + p(t)u\partial_x u)\, e_j \,\mathrm{d}x + q(t)\int_0^a \partial_x u \partial_x e_j \,\mathrm{d}x + \int_0^a r(t,x)\partial_x u e_j \,\mathrm{d}x \\ &= \int_0^a f e_j \,\mathrm{d}x,\end{aligned} \tag{2.14}$$

Since $(e_j)_{j\in\mathbb{N}}$ is a basis of $H_0^1(I)$, for all $w \in H_0^1(I)$, we can write

$$w(t) = \sum_{k=1}^{\infty} b_k(t) e_k,$$

that is to say $w_N(t) = \sum_{k=1}^{N} b_k(t)e_k \to w(t)$ in $H_0^1(I)$ when $N \to +\infty$.

Multiplying (2.14) by b_k and summing for $k = 1, \dots, N$, then

$$\begin{aligned}&\int_0^a (\partial_t u + p(t)u\partial_x u)\, w_N \,\mathrm{d}x + q(t)\int_0^a \partial_x u \partial_x w_N \,\mathrm{d}x + \int_0^a r(t,x)\partial_x u w_N \,\mathrm{d}x \\ &= \int_0^a f w_N \,\mathrm{d}x.\end{aligned}$$

Letting $N \to +\infty$, we deduce that

$$\int_0^a (\partial_t u + p(t)u\partial_x u)\, w \,\mathrm{d}x + q(t)\int_0^a \partial_x u \partial_x w \,\mathrm{d}x + \int_0^a r(t,x)\partial_x u w \,\mathrm{d}x = \int_0^a f w \,\mathrm{d}x,$$

so, u satisfies the weak formulation (2.3) for all $w \in H_0^1(I)$ and $t \in [0;T]$.

Finally, we recall that, by hypothesis, $\lim_{n\to+\infty} u_n(0) := u_0$. This completes the proof of the "existence" part of Theorem 2.1 . □

Lemma 2.8. *Under the assumptions of Theorem 2.1 , the solution of Problem* (2.1) *is unique.*

Proof. Let us observe that any solution $u \in H^{1,2}(R)$ of Problem (2.1) is in $L^\infty(0,T,L^2(I))$. Indeed, it is not difficult to see that such a solution satisfies

$$\frac{1}{2}\frac{d}{dt}\int_0^a u^2 \,\mathrm{d}x + q(t)\int_0^a (\partial_x u)^2 \,\mathrm{d}x - \frac{1}{2}\int_0^a \partial_x \gamma(t,x) u^2 \,\mathrm{d}x = \int_0^a f u \,\mathrm{d}x,$$

because

$$p(t)\int_0^a u^2\partial_x u\,\mathrm{d}x = \frac{p(t)}{3}\int_0^a \partial_x(u)^3\,\mathrm{d}x = 0,$$

and

$$\int_0^a r(t,x)\partial_x u u\,\mathrm{d}x = \int_0^a r(t,x)\partial_x(\frac{u^2}{2})\,\mathrm{d}x = -\frac{1}{2}\int_0^a \partial_x r(t,x)u^2\,\mathrm{d}x.$$

Consequently (see the proof of Lemma 2.4)

$$\begin{aligned}&\|u\|^2_{L^2(I)} + \alpha\int_0^t \|\partial_x u(s)\|^2_{L^2(I)}\,\mathrm{d}s\\ &\leq \|u_0\|^2_{L^2(I)} + \frac{a^2}{2\alpha}\int_0^t \|f(s)\|^2_{L^2(I)}\,\mathrm{d}s + \beta\int_0^t \|u(s)\|^2_{L^2(I)}\,\mathrm{d}s,\end{aligned}$$

so,

$$\begin{aligned}&\|u\|^2_{L^2(I)} + \alpha\int_0^t \|\partial_x u(s)\|^2_{L^2(I)}\,\mathrm{d}s\\ &\leq \|u_0\|^2_{L^2(I)} + \frac{a^2}{2\alpha}\int_0^t \|f(s)\|^2_{L^2(I)}\,\mathrm{d}s\\ &\quad + \beta\int_0^t \left(\|u(s)\|^2_{L^2(I)} + \alpha\int_0^s \|\partial_x u(\tau)\|^2_{L^2(I)}\,\mathrm{d}\tau\right)\,\mathrm{d}s.\end{aligned}$$

Then there exist a positive constant C such that

$$\begin{aligned}&\|u\|^2_{L^2(I)} + \alpha\int_0^t \|\partial_x u(s)\|^2_{L^2(I)}\,\mathrm{d}s\\ &\leq C + \beta\int_0^t \left(\|u(s)\|^2_{L^2(I)} + \alpha\int_0^s \|\partial_x u(\tau)\|^2_{L^2(I)}\,\mathrm{d}\tau\right)\,\mathrm{d}s.\end{aligned}$$

Hence, Gronwall's lemma gives

$$\|u\|^2_{L^2(I)} + \alpha\int_0^t \|\partial_x u(s)\|^2_{L^2(I)}\,\mathrm{d}s \leq K,$$

where $K = C\exp(\beta T)$. This shows that $u \in L^\infty(0,T,L^2(I))$ for all $f \in L^2(I)$.

Now, let $u_1, u_2 \in H^{1,2}(R)$ be two solutions of (2.1). We put $u = u_1 - u_2$. It is clear that $u \in L^\infty(0, T, L^2(I))$. The equations satisfied by u_1 and u_2 lead to

$$\int_0^a [\partial_t uw + \alpha(t)uw\partial_x u_1 + p(t)u_2 w\partial_x u + q(t)\partial_x u\partial_x w + r(t,x)w\partial_x u]\,\mathrm{d}x = 0.$$

Taking, for $t \in [0, T]$, $w = u$ as a test function, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I)} + \beta(t)\|\partial_x u\|^2_{L^2(I)}\\ &= -\int_0^a r(t,x)u\partial_x u\,\mathrm{d}x - p(t)\int_0^a u^2\partial_x u_1\,\mathrm{d}x - p(t)\int_0^a u_2 u\partial_x u\,\mathrm{d}x.\end{aligned} \tag{2.15}$$

An integration by parts gives

$$p(t)\int_0^a u^2\partial_x u_1\,\mathrm{d}x = -2p(t)\int_0^a u\partial_x u u_1\,\mathrm{d}x,$$

then (2.15) becomes

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I)} + q(t)\|\partial_x u\|^2_{L^2(I)} = \frac{1}{2}\int_0^a \partial_x r(t,x)u^2\,\mathrm{d}x + \int_0^a p(t)(2u_1 - u_2)u\partial_x u\,\mathrm{d}x.$$

By (2.2) and inequality (2.6) with $\varepsilon = 2\alpha$, it follows that

$$\begin{aligned}&|\int_0^a p(t)(2u_1 - u_2)u\partial_x u\,\mathrm{d}x|\\ &\leq \frac{\beta^2}{4\alpha}(2\|u_1\|_{L^\infty(I)} + \|u_2\|_{L^\infty(I)})^2\|u\|^2_{L^2(I)} + \alpha\|\partial_x u\|^2_{L^2(I)}.\end{aligned}$$

Then, using the injection of $H_0^1(I)$ in $L^\infty(I)$, we obtain

$$\begin{aligned}&|\int_0^a p(t)(2u_1 - u_2)u\partial_x u\,\mathrm{d}x|\\ &\leq \frac{\beta^2}{4\alpha}(2\|u_1\|_{L^\infty(0,T,H_0^1(I))} + \|u_2\|_{L^\infty(0,T,H_0^1(I))})^2\|u\|^2_{L^2(I)} + \alpha\|\partial_x u\|^2_{L^2(I)}.\end{aligned}$$

Furthermore,

$$\frac{1}{2}\int_0^a \partial_x\gamma(t,x)u^2\,\mathrm{d}x \leq \frac{\beta}{2}\|u\|^2_{L^2(I)}.$$

So,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I)}+\alpha\|\partial_x u\|^2_{L^2(I)}\\&\leq\frac{\beta^2}{4\alpha}(2\|u_1\|_{L^\infty(0,T,H_0^1(I))}+\|u_2\|_{L^\infty(0,T,H_0^1(I))})^2\|u\|^2_{L^2(I)}\\&\quad+\alpha\|\partial_x u\|^2_{L^2(I)}+\frac{\beta}{2}\|u\|^2_{L^2(I)}.\end{aligned}$$

We deduce that there exists a positive constant D, such that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I)}\leq D\|u\|^2_{L^2(I)},$$

and Gronwall's lemma leads to $u=0$. This completes the proof. □

2.2 Burgers equation in a domain that can be transformed into a rectangle

Let $\Omega\subset\mathbb{R}^2$ be the domain

$$\Omega=\{(t,x)\in\mathbb{R}^2: 0<t<T,\ x\in I_t\},$$
$$I_t=\{x\in\mathbb{R}:\varphi_1(t)<x<\varphi_2(t),\ t\in(0,T)\}.$$

In this section, we assume that $\varphi_1(0)\neq\varphi_2(0)$. In other words

$$\varphi_1(t)<\varphi_2(t)\quad\text{for all } t\in[0,T]. \tag{2.16}$$

and we consider the Burgers problem

$$\begin{cases}\partial_t u(t,x)+c(t)u(t,x)\partial_x u(t,x)-\partial_x^2 u(t,x)=f(t,x) & (t,x)\in\Omega,\\ u(0,x)=0\quad x\in I_0=(\varphi_1(0),\varphi_2(0)),\\ u(t,\varphi_1(t))=u(t,\varphi_2(t))=0\quad t\in(0,T),\end{cases} \tag{2.17}$$

in $\Omega\subset\mathbb{R}^2$, such that

$$c_1\leq c(t)\leq c_2,\quad\text{for all } t\in[0,T], \tag{2.18}$$

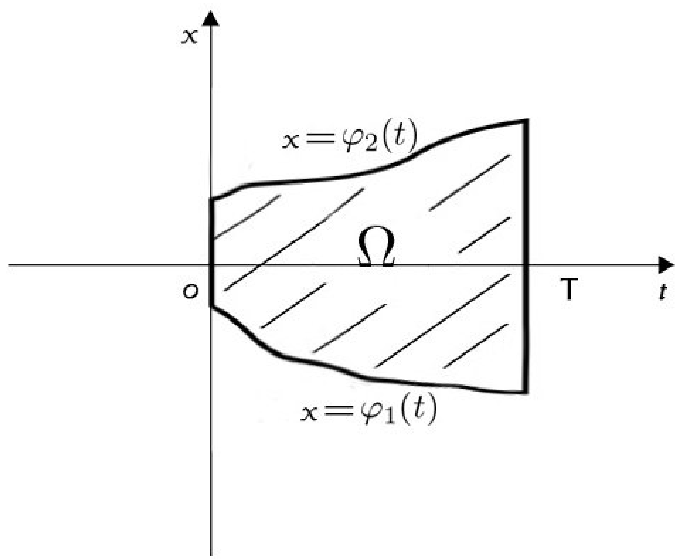

Figure 2.1: Domain that can be transformed into rectangle

where c_1 and c_2 are positive constants and φ_1, φ_2 are functions defined on $[0, T]$ belonging to $C^1(]0, T[)$.

Using the results obtained in the first part of this chapter, we look for conditions on the functions $(\varphi_i)_{i=1,2}$ which guarantee that Problem (2.17) admits a unique solution $u \in H^{1,2}(\Omega)$.

In order to solve Problem (2.17), we will follow the method which was used, for example, in Sadallah[42] and Clark *et al.* [14]. This method consists in proving that this problem admits a unique solution when Ω is transformed into a rectangle, using a change of variables preserving the anisotropic Sobolev space $H^{1,2}(\Omega)$.

To establish the existence and uniqueness of the solution to (2.17), we impose the assumption

$$|\varphi'(t)| \leq c \quad \text{for all } t \in [0, T] \tag{2.19}$$

where c is a positive constant, and $\varphi(t) = \varphi_2(t) - \varphi_1(t)$ for all $t \in [0, T]$.

The result related to the existence of the solution u of (2.17) in a rectangle is obtained thanks to a personal (and detailed) communication of professor Luc Tartar about the Burgers equation with constant coefficients in a rectangle. The authors would like to thank him for his appreciate comments and hints.

Theorem 2.9. *If $f \in L^2(\Omega)$ and $c(t)$, $(\varphi_i)_{i=1,2}$ satisfy the assumptions* (2.18), (2.16) *and* (2.19), *then Problem* (2.17) *admits a unique solution $u \in H^{1,2}(\Omega)$.*

The proof of Theorem 2.9 needs an appropriate change of variables which allows us to use Theorem 2.1 .

Proof. The change of variables: $\Omega \to R$

$$(t,x) \mapsto (t,y) = \left(t, \frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}\right)$$

transforms Ω into the rectangle $R = (0,T) \times (0,1)$. Putting $u(t,x) = v(t,y)$ and $f(t,x) = g(t,y)$, Problem (2.17) becomes

$$\begin{cases} \partial_t v(t,y) + p(t)v(t,y)\partial_y v(t,y) - q(t)\partial_y^2 v(t,y) + r(t,y)\partial_y v(t,y) \\ \qquad\qquad\qquad\qquad\qquad\qquad = g(t,y) \quad (t,y) \in R, \\ v(0,y) = 0 \quad y \in (0,1), \\ v(t,0) = v(t,1) = 0 \quad t \in (0,T), \end{cases} \tag{2.20}$$

where

$$\varphi(t) = \varphi_2(t) - \varphi_1(t), \quad p(t) = \frac{c(t)}{\varphi(t)},$$
$$q(t) = \frac{1}{\varphi^2(t)}, \quad r(t,y) = -\frac{y\varphi'(t) + \varphi_1'(t)}{\varphi(t)}.$$

This change of variables preserves the spaces $H^{1,2}$ and L^2. In other words

$$f \in L^2(\Omega) \;\Leftrightarrow\; g \in L^2(R),$$
$$u \in H^{1,2}(\Omega) \;\Leftrightarrow\; v \in H^{1,2}(R).$$

According to (2.18) and (2.19), the functions p, q and r satisfy the following conditions

$$\alpha < p(t) < \beta, \quad \forall t \in [0, T],$$

$$\alpha < q(t) < \beta, \quad \forall t \in [0, T],$$

$$|\partial_y r(t, y)| \leq \beta, \quad \forall (t, y) \in R,$$

where α and β are positive constants.

So, Problem (2.17) is equivalent to Problem (2.20), and by Theorem 2.1 Problem (2.20) admits a solution $v \in H^{1,2}(R)$. Then, Problem (2.17) in the domain Ω admits a solution $u \in H^{1,2}(\Omega)$. □