Chapter 1

Preliminaries and explicit solutions for non-homogeneous Burgers equations

This chapter consists of two parts, the objective of the first one is to recall the essential notions and the classical results that will be used throughout this work. In the second part, exact solutions for non-homogeneous Burgers equations with some choices of the second member are introduced, and the generalized Hopf-Cole transformation is used to transform the nonlinear Burgers equation into a linear heat equation.

1.1 Function Spaces

Functional analysis tools are the essential ingredients for the study of partial differential equations, especially Lebesgue and Sobolev spaces. The function spaces with vector values are adapted to the study of evolution problems. In this section, we will recall some fundamental results for the study of partial differential equations. For more details and proofs we refer to Adams [1], Brezis [10], Chipot [13] and Evans [20].

L^p spaces

Let Ω be an open subset of $\mathbb{R}^n$. $\mathcal{D}(\Omega)$ is the space of functions of class $\mathcal{C}^\infty$ with compact support. Let $p \in \mathbb{R}$ with $1 \leq p < \infty$, $f \in L^p(\Omega)$ if $f : \Omega \longleftrightarrow \mathbb{R}$ is measurable and $|f|^p$ is integrable. These spaces are equipped with the norm

$$\|f\|_{L^p(\Omega)} = \left(\int\limits_\Omega |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

$f \in L^\infty(\Omega)$ if f is measurable and there exists $C \geq 0$ such that $|f(x)| \leq C$ a.e on Ω. The space $L^\infty(\Omega)$ is equipped with the norm

$$\|f\|_{L^\infty(\Omega)} = \inf\{C \text{ such as } |f(x)| \leq C \ a.e \ x \text{ in } \Omega\}$$

Notation. Let $1 \leq p < \infty$; we denote by q the conjugate exponent of p,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Theorem 1.1. *(Hölder's inequality). Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p \leq \infty$. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

In the particular case where $p = q = 2$, we get Cauchy-Schwarz inequality:

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

Theorem 1.2. *$L^p(\Omega)$ is a Banach space for any p, $1 \leq p \leq \infty$ and it is a Hilbert space if $p = 2$.*

Theorem 1.3. *$D(\Omega)$ is dense in $L^p(\Omega)$.*

Reflexivity. Separability. Dual of L^p.

The following table summarizes the main properties of the space $L^p(\Omega)$ when Ω is a measurable subset of $\mathbb{R}^n$:

	Reflexive	Separable	Dual space
$L^p(\Omega)$ with $1 < p < \infty$	YES	YES	$L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$
$L^1(\Omega)$	NO	YES	$L^\infty(\Omega)$
$L^\infty(\Omega)$	NO	NO	$L^1(\Omega) \subsetneqq (L^\infty(\Omega))'$

Proposition 1.4. *Let (u_n) be a bounded sequence in $L^p(\Omega)$ with $1 < p < \infty$ and q such as $\frac{1}{p} + \frac{1}{q} = 1$, then we can extract from (u_n) a subsequence weakly convergent, i.e*

$$\exists (u_{n_k}), \exists u \in L^p(\Omega), \forall \varphi \in L^q(\Omega), \lim_{k \to +\infty} \int_\Omega u_{n_k} \varphi \, \mathrm{d}x = \int_\Omega u \varphi \, \mathrm{d}x.$$

Proposition 1.5. *Let (u_n) be a bounded sequence in $L^\infty(\Omega)$, then we can extract from (u_n) a subsequence weakly-star convergent, i.e*

$$\exists (u_{n_k}), \exists u \in L^\infty(\Omega), \forall \varphi \in L^1(\Omega), \lim_{k \to +\infty} \int_\Omega u_{n_k} \varphi \, \mathrm{d}x = \int_\Omega u \varphi \, \mathrm{d}x.$$

Sobolev spaces

Assume that Ω is an open domain in $\mathbb{R}^n$. For $m \in \mathbb{N}$ and $1 \le p \le \infty$. The Sobolev space $W^{m,p}(\Omega)$ is defined by:

$$W^{m,p}(\Omega) = \{ f \in L^p(\Omega);\ |\alpha| \le m, D^\alpha f \in L^p(\Omega) \},$$

where for any $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$ we note $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $D^\alpha f = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n} f = v$ is the $\alpha^{\text{th}}-$ weak derivative of f in the sense

$$\int_\Omega f D^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v \varphi \, \mathrm{d}x, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

The space $W^{m,p}(\Omega)$ is equipped with the norm

$$\|f\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \le m} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, \quad \text{if} \quad p < \infty,$$

and for $p = \infty$,

$$\|f\|_{W^{m,\infty}(\Omega)} = \sup_{|\alpha|\leq m} \|D^\alpha f\|_{L^\infty(\Omega)}.$$

Proposition 1.6.

i) $W^{m,p}(\Omega)$ *is a Banach space for every* $1 \leq p \leq \infty$.

ii) If $p < \infty$, $W^{m,p}(\Omega)$ *is separable.*

iii) If $1 < p < \infty$, $W^{m,p}(\Omega)$ *is reflexive.*

Proposition 1.7. *If* $u \in W_0^{m,p}(\Omega)$ *and* $\tilde{u}$ *is defined by*

$$\tilde{u} = \begin{cases} u & on \quad \Omega \\ 0 & on \quad \Omega^c \end{cases}$$

then $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$.

Notation 1.1. *If* $p = 2$, *we usually write* $W^{m,2}(\Omega) = H^m(\Omega)$. *The space* $H^m(\Omega)$ *is a separable Hilbert space. On the other hand* $\mathcal{D}(\overline{\Omega}) = \left\{u : \Omega \to \mathbb{C},\ \exists v \in \mathcal{D}(\mathbb{R}^n),\ u = v_{|\Omega}\right\}$.

Theorem 1.8. *If* Ω *is Lipschitz, the space* $\mathcal{D}(\overline{\Omega})$ *is dense in* $H^1(\Omega)$, *i.e. for any element* u *of* $H^1(\Omega)$ *there exists a sequence* (u_n) *of* $\mathcal{D}(\overline{\Omega})$ *such that* $\|u_n - u\|_{H^1(\Omega)}$ *converge to* 0.

Proposition 1.9. *If* Ω *is a bounded domain, then the semi-norm* $|\cdot|_{1,\Omega} : u \mapsto \|\nabla u\|_{L^2}$, *or simply* $\|\cdot\|_{H_0^1(\Omega)}$, *is a norm on* $H_0^1(\Omega)$ *which is equivalent to the norm induced by that of* $H^1(\Omega)$. *The space* $H_0^1(\Omega)$ *is a Hilbert space for the inner product defined by:*

$$(u, v) \mapsto (u, v)_{H_0^1(\Omega)} = (\nabla u, \nabla v)_{[L^2(\Omega)]^n}.$$

Theorem 1.10. *(compactness of Rellich-Kondrachov). Suppose that* Ω *is a bounded regular domain, then the embedding of* $H^1(\Omega)$ *in* $L^2(\Omega)$ *is compact. Consequently, all weakly convergent sequences in* $H^1(\Omega)$ *converge strongly in* $L^2(\Omega)$.

Theorem 1.11. *(Poincaré's inequality). If* Ω *is a bounded domain of* $\mathbb{R}^n$, *then there exists a constant* $C(\Omega)$ *such that*

$$\|u\|_{L^2(\Omega)} \leq C(\Omega)\|u\|_{H_0^1(\Omega)} \quad \forall u \in H_0^1(\Omega).$$

Theorem 1.12. *(of Lax-Milgram). Let H be a Hilbert space and $a(\cdot,\cdot)$ is a bilinear, continuous and coercive form on $H \times H$. Then, for any $\varphi \in H'$, there exists a unique $u \in H$ such that*

$$a(u,v) = \langle \varphi, v\rangle_{H',H}, \quad \forall v \in H.$$

Moreover, if $a : H \times H \to \mathbb{R}$ is symmetric, then u is characterized by the property:

$$u \in H \quad \text{and} \quad \frac{1}{2}a(u,u) - \langle \varphi, u\rangle_{H',H} = \min_{v\in H} \left\{ \frac{1}{2}a(v,v) - \langle \varphi, v\rangle_{H',H} \right\}.$$

1.1.1 Functional analysis for parabolic problems

Let X be a Banach space, with the norm $\|\cdot\|_X$.

Definition 1.13. *We denoted by $\mathcal{D}'(0,T;X)$ the space of distributions on $]0,T[$ with values in X, i.e. for $f \in \mathcal{D}'(0,T;X)$:*

i) $\phi \in \mathcal{D}(]0,T[;X) \mapsto \langle f, \phi\rangle$ *is linear.*

ii) for all ϕ_n of $\mathcal{D}(]0,T[;X)$ such that $\phi_n \to \phi$ in $\mathcal{D}(]0,T[;X)$, then $\langle f, \phi_n\rangle \to \langle f, \phi\rangle$ in X.

The convergence of the sequence of $\mathcal{D}'(0,T;X)$ is defined by:

$$f_n \to f \quad \text{in} \quad \mathcal{D}'(0,T;X) \quad \Longleftrightarrow \quad \forall \phi \in \mathcal{D}(0,T;X), \quad \langle f_n, \phi\rangle \to \langle f, \phi\rangle.$$

The derivative $\dfrac{\partial f}{\partial t}$ of $f \in \mathcal{D}'(0,T;X)$, is defined as the unique element of this space which satisfies

$$\langle \frac{\partial f}{\partial t}, \phi\rangle = -\langle f, \frac{\partial \phi}{\partial t}\rangle, \quad \forall \phi \in \mathcal{D}(]0,T[;X).$$

$L^p(0,T;X)$ spaces

Definition 1.14. *Let T be a positive number, for $p \in [1,+\infty[$, $L^p(0,T;X)$ denotes the space of classes of functions $f :]0,T[\longrightarrow X$ that are measurable, such that*

$$\int_0^T \|f(t)\|_X^p \,\mathrm{d}t < \infty,$$

It is a Banach space for the norm

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f(t)\|_X^p \right)^{\frac{1}{p}},$$

and for $p = +\infty$, $L^\infty(0,T;X)$ is a Banach space equipped with the norm

$$\|f\|_{L^\infty(0,T;X)} = \sup_{t\in]0,T[} ess\|f(t)\|_X.$$

Proposition 1.15. *If $p < \infty$, the set $C(0,T;X)$ of continuous functions of $[0,T]$ with values in X is dense in $L^p(0,T;X)$.*

Proposition 1.16. *For $p \in [1,\infty[$, we have the following results*

i) If X is separable, then $L^p(0,T;X)$ is also separable.

ii) If X is reflexive (respectively of Hilbert), then $L^p(0,T;X)$ is reflexive (respectively of Hilbert).

Duality.

Let p and q be two conjugate exponents, and $p \in [1,\infty[$.

i) The dual of $L^p(0,T;X)$ is $L^q(0,T;X')$.

ii) $L^p(0,T;L^p(\Omega)) = L^p(]0,T[\times\Omega)$.

Remark 1.17. *If X and Y are two Banach spaces such that*

$$X \hookrightarrow Y \quad (continuous\ embedding),$$

then

$$\mathcal{D}'(0,T,X) \hookrightarrow \mathcal{D}'(0,T,Y)$$

and

$$L^p(0,T,X) \hookrightarrow L^p(0,T,Y) \quad 1 \leq p \leq \infty.$$

For two Hilbert spaces V, H such that $V \hookrightarrow H \hookrightarrow V'$, V dense in H, where V' is the dual of V, a suitable choice could be

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega).$$

$W^{1,p}(0,T;X)$ spaces

Definition 1.18. *Let X be a Banach space and T a positive finite number. For $1 \leq p \leq \infty$, we define the space*

$$W^{1,p}(0,T;X) = \left\{ f \in L^p(0,T;X), \frac{\partial f}{\partial t} \in L^p(0,T;X) \right\}$$

equipped with the following norm

$$\|f\|_{W^{1,p}(0,T;X)} = \|f\|_{L^p(0,T;X)} + \left\| \frac{\partial f}{\partial t} \right\|_{L^p(0,T;X)}$$

or the equivalent norm

$$\|f\|_{W^{1,p}(0,T;X)} = \left(\|f\|^p_{L^p(0,T;X)} + \left\| \frac{\partial f}{\partial t} \right\|^p_{L^p(0,T;X)} \right)^{\frac{1}{p}}.$$

Proposition 1.19. *For $1 \leq p \leq \infty$, the space $W^{1,p}(0,T;X)$ is a Banach space.*

Proposition 1.20. *If X is separable (resp. reflexive) and $p < +\infty$, then the space $W^{1,p}(0,T;X)$ is separable (resp. reflexive) .*

For $p = 2$, we note $H^1(0,T;X) = W^{1,2}(0,T;X)$.

Proposition 1.21. *$H^1(0,T;X)$ is a Hilbert space.*

$H^1(Q)$ space

Definition 1.22. *Let Ω be an open of $\mathbb{R}^n$ and T a finite number. We note Q the cylinder defined by $Q =]0,T[\times\Omega$. We define the space*

$$H^1(Q) = \left\{ f \in L^2(Q), \frac{\partial f}{\partial t} \in L^2(Q) \right\}$$

equipped with the norm

$$\|f\|_{H^1(Q)} = \left(\|f\|^2_{L^2(Q)} + \left\| \frac{\partial f}{\partial t} \right\|^2_{L^2(Q)} \right)^{\frac{1}{2}}.$$

Theorem 1.23. *$H^1(Q)$ is a Hilbert space for the norm defined above.*

Theorem 1.24. *(of compactness). The embedding of $H^1(Q)$ in $L^2(Q)$ is compact.*

A general compactness result in vector-valued function spaces is given by the famous Lions-Aubin-Simon theorem

Theorem 1.25. *(Lions-Aubin-Simon) Let B_0, B and B_1 three Banach spaces with $B_0 \subset B \subset B_1$. We assume that the embedding $B_0 \hookrightarrow B$ is compact and that $B \hookrightarrow B_1$ is continuous. Let $1 < p < \infty$ and $1 < q < \infty$. We suppose that B_0 and B_1 are reflexive and we define*

$$W = \{f \in L^p(0, T, B_0),\ f' \in L^q(0, T, B_1)\},$$

W is a reflexive Banach space for the norm

$$\|f\|_W = \|f\|_{L^p(0,T,B_0)} + \|f'\|_{L^q(0,T,B_1)}$$

and the embedding $W \hookrightarrow L^p(0, T, B))$ is compact.

1.1.2 Anisotropic Sobolev spaces

In this section we introduce the so-called anisotropic Sobolev spaces $H^{r,s}$ built on the Lebesgue space of square integrable functions L^2. These function spaces are the natural ones adopted in the study of parabolic equations and are different from those in the study of elliptic equations since the space variable x and time variable t play different roles in parabolic equations. We recall the following definition of anisotropic Sobolev spaces. For more details we refer to [34]

Let r and s be two positive integers. For Ω an open set in $\mathbb{R}^n$, we define

$$H^{r,s}(Q) = L^2(0, T, H^r(\Omega)) \cap H^s(0, T, L^2(\Omega)), \quad (Q =]0, T[\times\Omega),$$

which is a Hilbert space with the norm

$$\left(\int_0^T \|u(t)\|^2_{H^r(\Omega)} \, \mathrm{d}t + \|u\|^2_{H^s(0,T,L^2(\Omega))} \right)^{\frac{1}{2}}$$

$H^r(\Omega)$ and $H^s(0, T, L^2(\Omega))$ are those defined previously.

Now, we give some basic properties of the anisotropic Sobolev space $H^{1,2}$.

The following result for the symmetric Sobolev space H^1 may be extended to anisotropic Sobolev space $H^{1,2}$.

Theorem 1.26. *Let Ω be a bounded open set with Lipschitz boundary and Ω_1, Ω_2 two open subsets of Ω with Lipschitz boundaries such that*

$$\overline{\Omega_1} \cup \overline{\Omega_2} = \overline{\Omega},$$
$$\Omega_1 \cap \Omega_2 = \varnothing.$$

Set $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. Let $u_1 \in H^1(\Omega_1)$, $u_2 \in H^1(\Omega_2)$ satisfying

$$u_1 = u_2, \quad on \quad \Gamma,$$

then the function u defined by

$$u = \begin{cases} u_1 \ in \ \ \Omega_1, \\ u_2 \ in \ \ \Omega_2, \end{cases}$$

belongs to $H^1(\Omega)$.

Proof. It is clear that $u \in L^2(\Omega)$. For an arbitrary $i \in \{1, 2, \cdots, n\}$ and a fixed $\varphi \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned} \left\langle \frac{\partial u}{\partial x_i}, \varphi \right\rangle &= -\left\langle u, \frac{\partial \varphi}{\partial x_i} \right\rangle \\ &= -\int_{\Omega_1} u_1 \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x - \int_{\Omega_2} u_2 \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x. \end{aligned}$$

For $k = 1, 2$

$$\int_{\Omega_k} u_k \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x = -\int_{\Omega_k} \frac{\partial u_k}{\partial x_i} \varphi \, \mathrm{d}x + \int_{\Gamma} u_k \varphi \nu_i^{(k)} \, \mathrm{d}x$$

because φ vanishes on $\partial\Omega_k \setminus \Gamma$, here $\nu^{(k)}$ is the outward normal vector on $\partial\Omega_k$. So, since $\nu^{(1)} = -\nu^{(2)}$ on Γ, we have

$$\left\langle \frac{\partial u}{\partial x_i}, \varphi \right\rangle = \int_{\Omega_1} \frac{\partial u_1}{\partial x_i} \varphi \, \mathrm{d}x + \int_{\Omega_2} \frac{\partial u_2}{\partial x_i} \varphi \, \mathrm{d}x + \int_{\Gamma} (u_2 - u_1) \varphi \nu_i^{(1)} \, \mathrm{d}x.$$

The boundary integral vanishs, so we obtain

$$\frac{\partial u}{\partial x_i} = \begin{cases} \dfrac{\partial u_1}{\partial x_i} \text{ in } \ \Omega_1, \\ \dfrac{\partial u_2}{\partial x_i} \text{ in } \ \Omega_2, \end{cases}$$

Since each $\dfrac{\partial u_k}{\partial x_i}$ belongs to $L^2(\Omega_k)$, we conclude that $\dfrac{\partial u}{\partial x_i} \in L^2(\Omega)$. □

1.1.3 Fixed point theorems

In nonlinear analysis, the fixed point theorems apply to justify the existence of a solution by a technique of transition to the nonlinear case from well-controlled linear cases. There are several fixed point theorems and their applications vary depending on the model and the hypotheses.

Theorem 1.27. *(Schauder fixed-point theorem). Let E be a normed vector space and C be a nonempty convex closed subset of E. If T is a continuous and compact mapping of C into itself, then T has a fixed point in C.*

Theorem 1.28. *(Tikhonov fixed-point theorem). Let $T : K \subseteq X \to K$ be a continuous application leaving invariant a nonempty, convex and compact set of a locally convex topological vector space. Then T admits a fixed point in K.*

Corollary 1.29. *(Schauder-Tikhonov fixed-point theorem). Let X a separable reflexive Banach space. We suppose that*

i) K is a nonempty, closed, bounded and convex set of X.

i) The application $T : K \to K$ is weakly sequentially continuous, i.e for any sequence (x_n) of K which converges weakly to x, when $n \to \infty$, the sequence $T(x_n)$ of K converges weakly to $T(x)$.

So, T admits at least one fixed point in K.

1.1.4 Technical Lemmas

Lemma 1.30. *(Young's inequality). Let p and q two positive reals such as $\frac{1}{p}+\frac{1}{q}=1$. For all positive reals a and b, we have*

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q}.$$

Theorem 1.31. *(Gronwall [25], p7), Let y, Ψ and χ be real continuous functions defined in $[a,b]$, $\chi(t)\geq 0$ for $t\in[a,b]$. We suppose that we have the inequality*

$$\forall t\in[a,b], \quad y(t)\leq \Psi(t)+\int_a^t \chi(s)y(s)\,\mathrm{d}s. \tag{1.1}$$

Then

$$y(t)\leq \Psi(t)+\int_a^t \chi(s)\Psi(s)\exp\left(\int_s^t \chi(u)\,\mathrm{d}u\right)\,\mathrm{d}s$$

in $[a,b]$.

Proof. Let us consider the function $g(t)=\int_a^t \chi(u)y(u)\,\mathrm{d}u$, $t\in[a,b]$. Then we have $g(a)=0$ and

$$\begin{aligned} g'(t) &= \chi(t)y(t) \\ &\leq \chi(t)\Psi(t)+\chi(t)\int_a^t \chi(s)y(s)\,\mathrm{d}s \\ &= \chi(t)\Psi(t)+\chi(t)g(t), \quad t\in(a,b). \end{aligned}$$

By multiplication with $\exp\left(-\int_a^t \chi(s)\,\mathrm{d}s\right)$, we obtain

$$\frac{d}{dt}\left(g(t)\exp\left(-\int_a^t \chi(s)\,\mathrm{d}s\right)\right)\leq \Psi(t)\chi(t)\exp\left(-\int_a^t \chi(s)\,\mathrm{d}s\right).$$

By integration on $[a,t]$, one gets

$$g(t)\exp\left(-\int_a^t \chi(s)\,\mathrm{d}s\right)\leq \int_a^t \Psi(u)\chi(u)\exp\left(-\int_a^u \chi(s)\,\mathrm{d}s\right)\,\mathrm{d}u,$$

so,

$$g(t) \leq \int_a^t \Psi(u)\chi(u) \exp\left(\int_u^t \chi(s)\,\mathrm{d}s\right) \mathrm{d}u, \quad t \in [a,b].$$

Since $y(t) \leq \Psi(t) + g(t)$, the theorem is thus proved. □

Next, we shall present some important corollaries resulting from Theorem 1.31.

Corollary 1.32. *If Ψ is differentiable, then from* (1.1) *it follows that*

$$y(t) \leq \Psi(a) \int_a^t \chi(u)\,\mathrm{d}u + \int_a^t \exp\left(\int_s^t \chi(u)\,\mathrm{d}u\right) \Psi'(s)\,\mathrm{d}s,$$

for all $t \in [a,b]$.

Proof. It is easy to see that

$$\begin{aligned}
&-\int_a^t \Psi(s) \frac{d}{dt}\left(\exp\left(\int_s^t \chi(u)\,\mathrm{d}u\right)\right) \mathrm{d}s \\
&= -\Psi(s) \exp\left(\int_s^t \chi(u)\,\mathrm{d}u\right)\Bigg|_a^b + \int_a^t \exp\left(\int_s^t \chi(u)\,\mathrm{d}u\right) \Psi'(s)\,\mathrm{d}s \\
&= -\Psi(t) + \Psi(a) \exp\left(\int_a^t \chi(u)\,\mathrm{d}u\right) + \int_a^t \exp\left(\int_s^t \chi(u)\,\mathrm{d}u\right) \Psi'(s)\,\mathrm{d}s
\end{aligned}$$

for all $t \in [a,b]$.

Hence,

$$\begin{aligned}
&\Psi(t) + \int_a^t \Psi(u)\chi(u) \exp\left(\int_u^t \chi(s)\,\mathrm{d}s\right) \mathrm{d}u \\
&= \Psi(a) \exp\left(\int_a^t \chi(u)\,\mathrm{d}u\right) + \int_a^t \exp\left(\int_s^t \chi(u)\,\mathrm{d}u\right) \Psi'(s)\,\mathrm{d}s, \quad t \in [a,b],
\end{aligned}$$

and the corollary is proved. □

Corollary 1.33. *If Ψ is constant, then from*

$$y(t) \leq \Psi + \int_a^t \chi(s) y(s)\,\mathrm{d}s.$$

it follows that

$$y(t) \leq \Psi \exp\left(\int_a^t \chi(s)\,\mathrm{d}s\right).$$

1.2 Exact solutions of nonhomogeneous Burgers equation

The nonlinear Burgers equation can be transformed into the linear heat equation and thus explicitly solved [27]. The linearization of the Burgers equation appeared in the twentieth century. It was discovered by Eberhard Hopf and Julian Cole, and named the Hopf-Cole Transformation in their honor. This transformation provides an interesting method for solving the viscous Burgers equation, and has also opened other doors for solving higher-order partial differential equations using similar methods.

The homogeneous Burgers equation $\partial_t u + u\partial_x u - \nu\partial_x^2 u = 0$, where ν represents the viscosity, is a model that has been solved explicitly, but only few specific cases have been solved for the nonhomogeneous Burgers equation $\partial_t u + u\partial_x u - \nu\partial_x^2 u = f(x,t)$. If the right-hand side depends only on time $f(x,t) = g(t)$, this equation can be transformed into an homogeneous Burgers equation in [39], the problem with $f(x,t) = kx$, $f(x,t) = \dfrac{kx}{(2\beta t+1)^2}$, $f(x,t) = g(t)x$ or with an elastic forcing term $f(x,t) = -k^2x + f(t)$ are discussed and analytical solutions are obtained in [52], [45], [38] and [44]. In different types of solutions of the forced Burgers equation with variable coefficients such as shock solitary wave, triangular wave, N -wave and rational function solutions are found and discussed.

1.2.1 Hopf-Cole transformation

In this part we consider the equation

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = f(x,t), \quad x \in \mathbb{R}, t > 0, \nu > 0, \tag{1.2}$$

with the initial data

$$u(x,0) = u_0(x). \tag{1.3}$$

The Burgers equation (1.2) is connected with the linear heat equation by the Hopf-Cole transformation

$$u = -2\nu \frac{\partial_x \varphi}{\varphi}. \tag{1.4}$$

From (1.4), we have

$$\begin{aligned}
\partial_t u &= \frac{2\nu(\partial_t\varphi\partial_x\varphi - \varphi\partial_t(\partial_x\varphi))}{\varphi^2},\\
u\partial_x u &= \frac{4\nu^2\partial_x\varphi(\varphi\partial_x^2\varphi - (\partial_x\varphi)^2)}{\varphi^3},\\
\nu\partial_x^2 u &= -\frac{2\nu^2(2(\partial_x\varphi)^3 - 3\varphi\partial_x\varphi\partial_x^2\varphi + \varphi^2\partial_x^3\varphi)}{\varphi^3}.
\end{aligned}$$

By substituting in Burgers equation (1.2), we obtain

$$\frac{2\nu(-\varphi\partial_t(\partial_x\varphi) + \partial_x\varphi(\partial_t\varphi - \nu\partial_x^2\varphi) + \nu\varphi\partial_x^3\varphi)}{\varphi^3} = f(x,t),$$

then

$$\partial_x\varphi\left(\partial_t\varphi - \nu\partial_x^2\varphi + F(x,t)\frac{\varphi}{2\nu}\right) = \varphi\left(\partial_t\varphi - \nu\partial_x^2\varphi + F(x,t)\frac{\varphi}{2\nu}\right)_x,$$

where

$$F(x,t) = \int f(x,t)dx + c(t).$$

Therefore, if φ solves the equation

$$\partial_t\varphi - \nu\partial_x^2\varphi = -F(x,t)\frac{\varphi}{2\nu} \tag{1.5}$$

then u solves Equation (1.2).

To completely transform the problem, we still have to see the initial condition. To do this, note that (1.4) can be written as follows

$$u = -2\nu\partial_x\left(ln\varphi\right),$$

then

$$\varphi(x,t) = ke^{\left(-\int\frac{u(x,t)}{2\nu}dx\right)}.$$

The initial condition (1.3) must therefore be transformed into

$$\varphi(x,0) = \varphi_0(x) = e^{\left(-\int \frac{u_0(x)}{2\nu}\,\mathrm{d}x\right)}.$$

So, we have reduced Equation (1.2) with the initial data (1.3) to the problem

$$\begin{cases} \partial_t\varphi - \nu\partial_x^2\varphi &= -F(x,t)\dfrac{\varphi}{2\nu}, \quad x\in\mathbb{R}, \quad t>0, \quad \nu>0, \\ \varphi(x,0) &= \varphi_0(x) = e^{\left(-\int \frac{u_0(x)}{2\nu}\,\mathrm{d}x\right)}, \quad x\in\mathbb{R}. \end{cases}$$

In what follows we are not interested in the uniqueness of the solution. So, we pass over the initial condition, and we try to find one of the solutions of the burgers equation.

1.2.2 Burgers equation with particular second member

In this section we obtain new exact solutions for the forced Burgers equation

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = f(t)x + g(t), \tag{1.6}$$

where f, g are arbitrary functions that depend on t, using the following transformation

$$\begin{gathered} u(x,t) = a(t)\frac{\partial_z v(\tau,z)}{v(\tau,z)} + b(t)x + c(t) \\ \tau = \tau(t), \quad z = \alpha(t)x + \beta(t) \end{gathered} \tag{1.7}$$

The functions $\alpha(t), \beta(t), \tau(t)$ and $z(x,t)$ are to determine later.
We have

$$\partial_t v(\tau,z) = \tau'\partial_\tau v + (\alpha' x + \beta')\partial_z v,$$

$$\partial_z(\partial_t v)(\tau,z) = \tau'\partial_z(\partial_\tau v) + (\alpha' x + \beta')\partial_z^2 v,$$

$$\partial_x v(\tau,z) = \alpha\partial_z v,$$

$$\partial_z(\partial_x v)(\tau,z) = \alpha\partial_z^2 v,$$

$$\partial_x(v^2)(\tau,z) = 2\alpha v\partial_z v,$$

$$\partial_x(\partial_z v^2)(\tau,z) = 2\alpha\partial_z v\partial_z^2 v,$$

then, we get

$$\partial_t u = a'\frac{\partial_z v}{v} + a\frac{(\alpha' x+\beta')\, v\partial_z^2 v + \tau' v\partial_z(\partial_\tau v) - (\alpha' x+\beta')\,\partial_z v^2 - \tau'\partial_\tau v\partial_z v}{v^2} + b'x + c',$$

$$\partial_x u = a\alpha\frac{v\partial_z^2 v - v_z^2}{v^2} + b,$$

$$u\partial_x u = a^2\alpha\frac{v\partial_z v\partial_z^2 v - (\partial_z v)^3}{v^3} + a\alpha(bx+c)\frac{v\partial_z^2 v - (\partial_z v)^2}{v^2} + ab\frac{\partial_z v}{v} + b^2 x + bc,$$

and

$$\partial_x^2 u = a\alpha^2\frac{v\partial_z^3 v - 3\partial_z v\partial_z^2 v}{v^2} + 2a\alpha^2\frac{(\partial_z v)^3}{v^3}.$$

So, Equation (1.6), can be written as follows

$$\begin{aligned}
&a\tau'\frac{\partial_z(\partial_\tau v)}{v} - \nu a\alpha^2\frac{\partial_z^3 v}{v} + (a^2\alpha + 3\nu a\alpha^2)\frac{\partial_z v\partial_z^2 v}{v^2} - a\tau'\frac{\partial_\tau v\partial_z v}{v^2} + (a' + ab)\frac{\partial_z v}{v}\\
&- (a^2\alpha + 2\nu a\alpha^2)\frac{(\partial_z v)^3}{v^3} - a\left(\alpha bx + \alpha c + \alpha' x + \beta'\right)\frac{(\partial_z v)^2}{v^2} + a\left(\alpha bx + \alpha c + \alpha' x + \beta'\right)\frac{\partial_z^2 v}{v}\\
&+ b^2 x + bc + b'x + c' - f(t)x - g(t) = 0.
\end{aligned}$$

Then

$$\begin{aligned}
&a\tau'\frac{\partial_z(\partial_\tau v)}{v} - \nu a\alpha^2\frac{\partial_z^3 v}{v} - a\tau'\frac{\partial_\tau v\partial_z v}{v^2} + \nu a\alpha^2\frac{\partial_z v\partial_z^2 v}{v^2}\\
&+ (a^2\alpha + 2\nu a\alpha^2)\frac{\partial_z v\partial_z^2 v}{v^2} + (a' + ab)\frac{\partial_z v}{v} - (a^2\alpha + 2\nu a\alpha^2)\frac{(\partial_z v)^3}{v^3}\\
&- a\left((\alpha b + \alpha')x + \alpha c + \beta'\right)\frac{(\partial_z v)^2}{v^2} + a\left((\alpha b + \alpha')x + \alpha c + \beta'\right)\frac{\partial_z^2 v}{v}\\
&+ ((b^2 + b' - f(t))x + bc + c' - g(t) = 0.
\end{aligned}$$

by considering the following conditions

$$\begin{aligned}
a\tau'(t) - \nu a\alpha^2 &= 0,\\
\alpha' + \alpha b &= 0,\\
\beta' + \alpha c &= 0,\\
bc + c' - g &= 0,\\
b^2 + b' - f(t) &= 0,
\end{aligned}$$

we obtain

$$a\tau'\frac{\partial_z(\partial_\tau v)}{v} - \nu a\alpha^2\frac{\partial_z^3 v}{v} - a\tau'\frac{\partial_\tau v\partial_z v}{v^2} + \nu a\alpha^2\frac{\partial_z v\partial_z^2 v}{v^2} = 0.$$

Putting $a^2\alpha + 2\nu a\alpha^2 = 0$, we find that v satisfies

$$\partial_z v(\partial_z v - \nu\partial_z^2 v) - v\partial_z(\partial_z v - \nu\partial_z^2 v) = 0,$$

which yields to the linear heat equation,

$$\partial_\tau v - \nu\partial_z^2 v = 0.$$

The unknown functions satisfy the following system of ordinary differential equations

$$\begin{cases} \tau' = \nu\alpha^2(t), \\ a = -2\nu\alpha, \\ \alpha' = -\alpha b, \\ \beta' = -\alpha c, \\ c' = g - bc. \end{cases}$$

To solve this system we start with the third equation

$$\begin{aligned} \alpha(t) &= c_1 \exp\left(-\int b(t)dt\right), \\ \beta(t) &= -\int \alpha(t)c(t)dt + c_2, \\ \tau(t) &= \nu\int \alpha^2(t)dt + c_3, \\ a(t) &= -2\nu\alpha(t), \\ c(t) &= \alpha(t)\int \frac{g(t)}{\alpha(t)}dt - c_4\alpha(t), \end{aligned}$$

here c_1, c_2, c_3 and c_4 are arbitrary constants.

Then, solving Equation (1.6) is equivalent to solve the following two equations

$$\partial_z v = \nu\partial_z^2 v, \quad b'(t) = -b^2(t) + f(t).$$

We note that v satisfies the heat equation, and b is a solution of the Riccati equation.

Example 1.34. *For $f(t) = g(t) = 1$, Equation (1.6) becomes*

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = x + 1.$$

By substituting for example $v(z,\tau) = z^2 + 2\nu\tau$ *as a solution of the linear heat equation into Transformation* (1.7)*, we can get for the precedent equation the following solution*

$$u(x,t) = 2\nu e^{-t} \frac{e^{-t} - 2x - 1}{\left(x - \frac{e^{-t}}{2} + 1\right)^2 - \nu} + x + 1 - e^{-t}.$$

1.2.3 Burgers equation with other second member

In this section we present exact solutions for the following forced Burgers equations

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = f(t), \tag{1.8}$$

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = g(x), \tag{1.9}$$

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = \alpha e^{\alpha x + \beta t}, \tag{1.10}$$

where f, is a function that depends on t, and g is a function that depends on x.

By the Hopf-Cole transformation

$$u = -2\nu\frac{\partial_x\varphi}{\varphi},$$

if φ solve equations

$$\varphi - \nu\partial_x^2\varphi + \frac{1}{2\nu}xf(t)\varphi = 0, \tag{1.11}$$

$$\varphi - \nu\partial_x^2\varphi + G(x)\varphi = 0, \quad (\text{where } G(x) = \int g(x)\,\mathrm{d}x) \tag{1.12}$$

$$\varphi - \nu\partial_x^2\varphi + \frac{1}{2\nu}e^{\alpha x + \beta t}\varphi = 0, \tag{1.13}$$

then u solve Equations (1.8), (1.9) and (1.10).

The transformations that will be used in the following are cited in [40].

a) Equation (1.11)

Let the transformation

$$\varphi(x,t) = w(z,t)\exp\left(xF(t) + \nu\int F^2(t)dt\right),$$

$$z = x + 2\nu\int F(t)dt,$$

where $F(t) = -\dfrac{1}{2\nu}\displaystyle\int f(t)dt$.
We have

$$\begin{aligned}
\partial_t\varphi &= \left(\partial_t w + \partial_z w \partial_t z + \left(-\frac{1}{2\nu}xf(t) + \nu F^2(t)\right) w\right) \exp\left(xF(t) + \nu \int F^2(t)dt\right) \\
&= \left(\partial_t w + 2\nu F(t)\partial_z w + \left(-\frac{1}{2\nu}xf(t) + \nu F^2(t)\right) w\right) \exp\left(xF(t) + \nu \int F^2(t)dt\right), \\
\partial_x\varphi &= \left(\partial_z w \partial_x z + F(t)w\right) \exp\left(xF(t) + \nu \int F^2(t)dt\right) \\
&= \left(\partial_z w + F(t)w\right) \exp\left(xF(t) + \nu \int F^2(t)dt\right),
\end{aligned}$$

and

$$\begin{aligned}
\partial_x^2\varphi &= \left(\partial_z^2 w + F(t)\partial_z w + F(t)\left(\partial_z w + F(t)w\right)\right) \exp\left(xF(t) + \nu \int F^2(t)dt\right) \\
&= \left(\partial_z^2 w + 2F(t)\partial_z w + F^2(t)w\right) \exp\left(xF(t) + \nu \int F^2(t)dt\right).
\end{aligned}$$

Injecting the precedent results into (1.11), we obtain

$$\partial_t w + 2\nu F(t)\partial_z w - \frac{1}{2\nu}xf(t)w + \nu F^2(t)w - \nu\partial_z^2 w - 2\nu F(t)\partial_z w - \nu F^2(t)w + \frac{1}{2\nu}xf(t)w = 0,$$

which leads to the heat equation

$$w_t - \nu\partial_z^2 w = 0.$$

b) Equation (1.12)

There are particular solutions in the product form

$$\varphi(x,t) = e^{\lambda t}w(x),$$

where λ is an arbitrary constant and w is determined an ordinary differential equation.

In fact,

$$\lambda e^{\lambda t}w - \nu e^{\lambda t}w'' + G(x)e^{\lambda t}w = 0,$$

then

$$\lambda w'' + (G(x) + \lambda)w = 0.$$

Remark 1.35. *From the above we can obtain a particular solution for the Burgers equation when the second member is $f(t) + g(x)$,*

c) Equation (1.13)

From the transformation

$$\varphi(x,t) = w(z,t)e^{\mu x}, \quad z = x + \frac{\beta}{\alpha}t \ \text{ where } \ \mu = \frac{\beta}{2\nu\alpha}.$$

We have

$$\begin{aligned}
\partial_t \varphi &= \left(\partial_z w \partial_t z + \partial_t w\right) e^{\mu}(z + \tfrac{\beta}{\alpha}t) \\
&= \left(\frac{\beta}{\alpha}\partial_z w + \partial_t w\right) e^{\mu}(z + \tfrac{\beta}{\alpha}t), \\
\partial_x \varphi &= \left(\partial_z w \partial_x z + \mu w\right) e^{\mu}(z - \tfrac{\beta}{\alpha}t) \\
&= \left(\partial_z w + \mu w\right) e^{\mu}(z - \tfrac{\beta}{\alpha}t),
\end{aligned}$$

and

$$\begin{aligned}
\partial_x^2 \varphi &= \left(\partial_x(\partial_z w) + \mu\partial_x w + \mu\partial_x w + \mu^2 w\right) e^{\mu}(z - \tfrac{\beta}{\alpha}t) \\
&= \left(\partial_z^2 w + 2\mu\partial_z w + \mu^2 w\right) e^{\mu}(z - \tfrac{\beta}{\alpha}t).
\end{aligned}$$

Submitting into (1.11), we obtain

$$\partial_t w - \nu\partial_z^2 w + \left(\frac{1}{2\nu}e^{\alpha z} - 3\nu\mu^2\right) w = 0.$$

witch is of the form (1.12) where $G(z) = \dfrac{1}{2\nu}e^{\alpha z} - 3\nu\mu^2$.