Introduction

One of the most important partial differential equations of the theory of nonlinear conservation laws, is the semilinear diffusion equation, called Burgers equation:

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = f, \tag{1}$$

where u stands, generally, for a velocity, t the time variable, x the space variable and ν the constant of viscosity (or the diffusion coefficient). Homogeneous Burgers equation ((1) with $f = 0$), is one of the simplest models of nonlinear equations which have been studied.

The mathematical structure of this equation includes a nonlinear convection term $u\partial_x u$ which makes the equation more interesting, and a viscosity term of higher order $\partial_x^2 u$ which regularizes the equation and produces a dissipation effect of the solution near a shock.

When the viscosity coefficient vanishes, $\nu = 0$, the Burgers equation is reduced to the transport equation, which represents the inviscid Burgers equation $\partial_t u + u\partial_x u = f$.

The study of the equation (1) has a long history: In 1906, Forsyth, treated an equation which converts by some variable changes to the Burgers equation. In 1915, Bateman [3] introduced the equation (1): He was interested in the case when $\nu \to 0$, and in studying the movement behavior of a viscous fluid when the viscosity tends to zero. Burgers (1948) has published a study on the equation (1) (which it owes his name), in his document [12] about modeling the turbulence phenomena. Using the transformation discovered later by Cole [16] in 1951, about the same time and independently by Hopf [27], (called the Hopf-Cole transformation), Burgers continued his study of what he called "nonlinear diffusion equation". This study treated mainly the static aspects of the equation. The results of these works can be found in the book [11].

The objective of Burgers was to consider a simplified version of the incompressible Navier Stokes equation $\partial_t u + (u \cdot \nabla)u = \nu \Delta u - \nabla p$ by neglecting the pressure term.

Among the most interesting applications of the one-dimensional Burgers equation, we mention traffic flow, growth of interfaces, and financial mathematics (see for example [30, 50]).

The nonlinear Burgers equation (1), with $f = 0$, can be converted to the linear heat equation and then explicitly solved by the Hopf-Cole transformation.

For $f(x,t) = -\lambda \partial_x \eta(x,t)$, Burgers equation becomes

$$\partial_t u + u\partial_x u - \nu \partial_x^2 u = -\lambda \partial_x \eta(x,t), \tag{2}$$

which is Burgers stochastic equation, where $\eta(x,t)$ stands for the white noise and λ is a positive constant. Using the transformation $u(x,t) = -\lambda \partial_x h(x,t)$, we find that (2) is equivalent to the equation of Kardar, Parisi and Zhang (KPZ equation)

$$\partial_t h(x,t) - \frac{\lambda}{2}(\partial_x h(x,t))^2 - \nu \partial_x^2 h(x,t) = \eta(x,t).$$

This equation has been introduced by Kardar, Parisi and Zhang in 1986, and quickly became the default model for random interface growth in physics.

In this work we are interested in proving a result of existence, uniqueness and regularity for the non homogeneous Burgers problem. On the other hand we are looking for explicit solutions for what is also called the forced Burgers equation $\partial_t u + u\partial_x u - \nu \partial_x^2 u = f$, where $f(x,t)$ is the forcing term in a rectangular domain.

In [42], the author studied a linear parabolic equation in a domain similar to the one considered in this thesis. Other references on the analysis of linear parabolic problems in non-regular domains are discussed for example in [2, 24, 31, 32].

The work by Clark et al. [14] is devoted to the homogeneous Burgers equation in domains which can be transformed into rectangle. In the same domains, we have established the existence, uniqueness and the optimal regularity of the solution to the non-homogeneous Burgers equation with time variable coefficients in an anisotropic Sobolev space [4]. Then we have extend this last work to another type of non parabolic domain [5].

For Korteweg-de Vries (KdV) equation, the propagation of one directional long water waves of a small and finite amplitude on a channel, can be described as is well known, by the KdV equation (see [22, 35]):

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) + \partial_x^3 u(t,x) = 0,$$

where u represents the wave propagation, t the time variable and x the space variable. This equation expresses a balance between dispersion form from its third derivative term $\partial_x^3 u$, and the shock forming tendency of its nonlinear term $u\partial_x u$. This equation is a mathematical model known as a prototype example of nonlinear equations that have been explicitly solved.

Zabusky and Kruskal [54] have discovered that the solutions of KdV equation can present solitons, which are solitary waves that propagate without deforming in a non-linear and dispersive medium.

Korteweg and de Vries have published a study on the KdV equation in 1895 concerning the description of solitary wave phenomenon discovered by Russel [41], which initially observed while walking along a channel: "He followed for several kilometers a wave remounting the flow, which does not weaken". However, the first application of the KdV theory was actually made a few years before the publication of KdV equation in 1895, as have pointed out by Bullough and Caudrey [8]. Later, it was established that the KdV equation occurs in a large variety of physical situations, for example in the case of magneto-acoustic wave, or of ionic wave (plasma wave). It also can be applied to the acoustic structures of the dust in dusty plasmas magnetized [28], to sound waves and to shallow water waves. Other applications of the equation such as gravity waves, internal solitons in the ocean, nonlinear acoustics of bubbling liquids and others, are well examined by Crighton [17].

When the surface of fluid is submitted to a non constant pressure, or when the bottom of layer is not flat, a forcing term has to be added to the equation (see [51]). In the absence of the forcing term Miura, Gardner and Kruskal [36], have showed that the solutions of the KdV equation can be found analytically [26, 48].

The existence of a solution for the homogeneous initial value problem of Korteweg-de Vries has been proved by Bona and Smith [7], Kametaka [29], Sjoberg [43], Tsutsumi [49], Temam [47], Doronin and Larkin [18], [33] . However, in our work we are interested in proving a result of existence, uniqueness and regularity for the non-homogeneous Korteweg-de Vries problem with variable coefficients in non-rectangular domains which can be reduced to rectangular domains by changes of regular variables.

Let us briefly indicate the contents of every chapter of this thesis.

The first part of CHAPTER 1 is an overview. In the second one, we consider a nonhomogeneous Burgers equation of the form

$$\partial_t u + u\partial_x u - \partial_x^2 u = f(t)x + g(t), \tag{3}$$

where f and g are arbitrary functions that depend on t and we obtain a new exact solutions thanks to the generalized Hopf-Cole transformation. We also present exact solutions when the right-hand side of (3) is $f(t)$, $g(x)$ and $e^{\alpha x+\beta t}$.

CHAPTER 2 is concerned with two questions regarding the Burgers equation. The first one is to study the semilinear parabolic problem

$$\begin{cases} \partial_t u(t,x) + p(t)u(t,x)\partial_x u(t,x) - q(t)\partial_x^2 u(t,x) + r(t,x)\partial_x u(t,x) = f(t,x) \ (t,x) \in R, \\ u(0,x) = u_0(x) \quad x \in I, \\ u(t,a) = u(t,0) = 0 \quad t \in (0,T), \end{cases}$$

in the rectangle $R = (0,T) \times I$ where $I = (0,a)$, $a \in \mathbb{R}^+$ (T is finite); $f \in L^2(R)$ and $u_0 \in H_0^1(I)$ are given functions.

We assume that the functions p, q depend only on t and the function r depends on t and x. We also suppose that there exist positive constants α and β , such that

$$\alpha \le p(t) \le \beta, \qquad \alpha \le q(t) \le \beta, \qquad \forall t \in (0,T)$$
$$\textit{and} \qquad |\partial_x r(t,x)| \le \beta \quad \text{where} \quad |r(t,x)| \le \beta \qquad \forall (t,x) \in R.$$

The second question concerns the semilinear parabolic Burgers problem

$$\begin{cases} \partial_t u(t,x) + c(t)u(t,x)\partial_x u(t,x) - \partial_x^2 u(t,x) = f(t,x) \quad (t,x) \in \Omega, \\ u(0,x) = 0 \quad x \in I_0 = (\varphi_1(0), \varphi_2(0)), \\ u(t,\varphi_1(t)) = u(t,\varphi_2(t)) = 0 \quad t \in (0,T), \end{cases}$$

in $\Omega \subset \mathbb{R}^2$, where

$$\Omega = \{(t,x) \in \mathbb{R}^2 : 0 < t < T,\ x \in I_t\},$$
$$I_t = \{x \in \mathbb{R} : \varphi_1(t) < x < \varphi_2(t),\ t \in (0,T)\}.$$

$f \in L^2(\Omega)$ and $c(t)$ is given. φ_1, φ_2 are functions defined on $(0,T)$ belonging to $C^1((0,T))$, and such that

$$\varphi_1(t) < \varphi_2(t) \quad \forall t \in [0,T].$$

We assume that

$$|\varphi'(t)| \leq c \quad \forall t \in [0,T],$$

where c is a positive constant, and $\varphi(t) = \varphi_2(t) - \varphi_1(t)$.

Then, the Burgers problem admits a unique solution in the anisotropic Sobolev space

$$H^{1,2}(\Omega) = \{u \in L^2(\Omega); \partial_t u, \partial_x u, \partial_x^2 u \in L^2(\Omega)\}.$$

Result's publication: Y. Benia, B-K. Sadallah: *Existence of solutions to Burgers equations in domains that can be transformed into rectangles*, Electron. J. Diff. Equ., **157** (2016), 1-13.

In Chapter 3, we study the Burgers equation with time variable coefficients, subject to Dirichlet boundary condition. In Section 3.2 (resp. Section 3.3), we solve the problem

$$\begin{cases} \partial_t u(t,x) + c(t)u(t,x)\partial_x u(t,x) - \partial_x^2 u(t,x) = f(t,x) \quad (t,x) \in \Omega, \\ u(t,\varphi_1(t)) = u(t,\varphi_2(t)) = 0 \quad t \in (0,T), \end{cases}$$

when φ_1 and φ_2 are monotone on $(0,T)$ (resp. near 0), in a non-parabolic domain

$$\Omega = \{(t,x) \in \mathbb{R}^2 : 0 < t < T,\ x \in I_t\},$$
$$I_t = \{x \in \mathbb{R} : \varphi_1(t) < x < \varphi_2(t),\ t \in]0,T]\}.$$

with the fundamental hypothesis

$$\varphi_1(0) = \varphi_2(0).$$

The most interesting point of the problem studied in this chapter is the fact that φ_1 allowed to coincide with φ_2 for $t = 0$, which prevents the domain is not rectangular and cannot be transformed into a regular domain without the appearance of some degenerate terms in the equation. In order to overcome this difficulty, we assume that φ_1 and φ_2 are monotone on $(0, T)$, so that Burgers problem admits a unique solution in the anisotropic Sobolev space $H^{1,2}(\Omega)$.

We approximate this domain by a sequence of subdomains $(\Omega_n)_{n\in\mathbb{N}}$. Then we establish an a priori estimate of the type

$$\|u_n\|^2_{H^{1,2}(\Omega_n)} \leq K\|f\|^2_{L^2(\Omega)},$$

where u_n is the solution of problem considered in Ω_n and K is a constant independent of n. This inequality allows us to pass to the limit in n.

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In Chapter 4, we study the semi-linear Korteweg-de Vries problem with time variables coefficients

$$\begin{cases} \partial_t v(t,y) + c(t)v(t,y)\partial_y v(t,y) + \partial_y^3 v(t,y) = g(t,y) \quad \text{in } \Omega, \\ v(0,y) = 0, \quad y \in I_0, \\ v(t,\varphi_1(t)) = v(t,\varphi_2(t)) = \partial_y v(t,\varphi_2(t)) = 0 \quad t \in (0,T), \end{cases}$$

where $I_0 = (\varphi_1(0), \varphi_2(0))$ and $a(t)$, $b(t)$ are given. g and $\partial_t g$ are in $L^2(\Omega)$.

We assume that there exist positive constants c_1 and c_2, such that

$$c_1 \leq c(t) \leq c_2, \quad \text{for all } t \in (0,T),$$
$$c_1 \leq c'(t) \leq c_2, \quad \text{for all } t \in (0,T).$$

We also impose, the hypothesis

$$\varphi_1',\ \varphi_2',\ \varphi_1''\ , \varphi_2''\ , \varphi_1'''\ , \varphi_2''' \in L^1(0,T).$$

Then, we establish the existence and the uniqueness of a solution in the anisotropic space

$$H^{1,3}(\Omega) = \{u \in L^2(\Omega); \partial_t u, \partial_x u, \partial_x^2 u, \partial_x^3 u \in L^2(\Omega)\}.$$

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