CHAPTER 5

On the regularity of the heat equation solution in non-cylindrical domains: two approaches

Abstract. In this work, we investigate the behavior of the solution of the Cauchy-Dirichlet problem for a parabolic equation set in a three-dimensional domain with edges. We also give new regularity results for the weak solution of this equation in terms of the regularity of the initial data.

Key words. Parabolic equations, heat equation, polyhedral domains, edges, anisotropic Sobolev spaces, interpolation theory.

5.1 Introduction

At the present time there exists a comprehensive theory of boundary value problems for parabolic equations and systems with a smooth boundary. One of the central results of this theory consists in the fact that if the coefficients of the equation and of the boundary operators, their right-hand sides, and the boundary of the domain are sufficiently smooth (the initial and boundary conditions must also satisfy the so-called compatibility conditions), then the solution itself of the problem is correspondingly smooth, see [23], [1], [43] and [35]. The lack of boundary smoothness in these problems leads to the occurrence of singularities of the solution in the neighbourhood of non-regular points of the boundary.

It is well known that there are two main approaches for the study of boundary value problems in non-regular domains. We can impose conditions on the non-regular domains to obtain regular solutions (see, for example [58]), or we work directly in the non-regular domains and we obtain singular solutions. The second approach will be illustrated in this work by the analysis of the heat equation in a domain of $\mathbb{R}^3$ with edge.

The first part of this work is concerned with the extension of solvability results for a parabolic equation, set in a non-convex polygon obtained in [60], to the case of a polyhedral domain with edge on the boundary. In a previous work [26], we have proved that under some conditions on the functions of the parametrization of a three-dimensional domain, the solution of the heat equation is "regular". The domains considered there include all the convex polyhedral domains (see, Sadallah [59]), but not all the polyhedral domains.

Let G be a non-convex bounded polyhedral domain of $\mathbb{R}^3$. In G, we consider the boundary value problem

$$\begin{cases} \partial_t u - \partial_x^2 u - \partial_y^2 u = f \\ \\ u_{/\partial_p G} = 0, \end{cases} \tag{5.1.1}$$

where $\partial_p G$ is the parabolic boundary of G and $f \in L^2(G)$. From now on, the parabolic operator $\partial_t - \partial_x^2 - \partial_y^2$ will be denoted by L.

The solvability of this kind of problems in the case of one-dimensional space variable has been investigated, for instance, in Aref'ev and Bagirov [4], [5] and in Sadallah [60] where results concerning the behavior of the solution of the heat equation in various singular domains of $\mathbb{R}^2$ were obtained. Solvability results for parabolic equations in domains with edges can be found in [13] and [54]. The solutions of boundary value problems, when posed and solved in non-smooth domains like polygons and polyhedra, have singular parts which are described in terms of special functions depending on the geometry of the domain and on the differential operators (see, for example, [20], [21] and the references therein).

The aim of the first approach is two-fold: Firstly, we exhibit singularities which appear in the solution u of Problem (5.1.1). Secondly, we study their smoothness. More precisely, we prove that there exist two functions v and w such that $u = v + w$ where v belongs to the anisotropic Sobolev space

$$H^{1,2}(G) = \left\{v \in L^2(G) : \partial_t v, \partial_x^j v, \partial_y^j v, \partial_{xy}^2 v \in L^2(G), j = 1, 2\right\}$$

whereas the singular part w is in the space $H^{r,2r}(G)$ with $r < 3/4$, defined as an interpolation space between $H^{1,2}(G)$ and $L^2(G)$, (the Sobolev spaces $H^{r,2r}(G)$ are those defined in Lions and Magenes [43]).

Our interest in the second approach is the regularity of the solutions of the heat equation posed in a non-cylindrical domain - subject to Dirichlet conditions on the lateral boundary- in terms of the regularity of the inhomogeneous initial Cauchy data.

The plan of this chapter is as follows. In Section 5.2, we begin by preliminaries where we define the non-convex polyhedral domain and the basic functional spaces, in which we will work. Then, we describe the asymptotic behavior of the solution in the neighborhood of an edge in a model domain which is the union of two parallelepipeds. We will show that the solution may be written as a sum of a function which is the solution of a problem of type (5.1.1) and an infinite number of functions which are solutions of an homogeneous problem related to the problem (5.1.1). The main result concerning the optimal regularity of the singular part w is presented in Theorem 5.3.2, that is,

$$w \in H^{r,2r}(G) \Longleftrightarrow r < 3/4.$$

The proof is based on the Fourier transform as well as on some properties of interpolation theory and the fractional powers of operators. In Section 5.3, by using some interpolation results we prove the regularity of the weak solution of the heat equation set in a cylindrical domain in terms of the initial data. Finally, we use this case to prove new results of regularity of the weak solutions of the heat equation in a domain which is the union of two cylinders.

5.2 First approach

5.2.1 Description of the domain

In this section we introduce some notations defining the non-convex polyhedral domain in which we will work. As a model, we choose a domain Q, which is the union of two parallelepipeds, see Fig. 8. Q is the simplest polyhedral domain which guarantees the appearance of the singular part in the solution of Problem (5.1.1). Hereafter, some notations in a coordinates system of variables (t, x, y):

$Q = Q_1 \cup Q_2 \cup \Omega$, where $Q_1 =]-1,0[\times]-1,1[\times]0,1[$, $Q_2 =]0,1[^3$ and $\Omega = \{0\} \times]0,1[\times]0,1[$.

Let us now give some notations concerning the boundary (faces and edge) of the polyhedral domain defined above.

$\Omega_1 = \{-1\} \times]-1,1[\times]0,1[$, $\Omega_1' =]-1,0[\times \{1\} \times]0,1[\cup]-1,0[\times \{-1\} \times]0,1[$, $\Omega_1'' = \{0\} \times]-1,0[\times]0,1[$, $\Omega_2' =]0,1[\times \{0\} \times]0,1[\cup]0,1[\times \{1\} \times]0,1[$, $\Omega_2'' = \{1\} \times]0,1[\times]0,1[$, $A = \{0\} \times \{0\} \times]0,1[$.

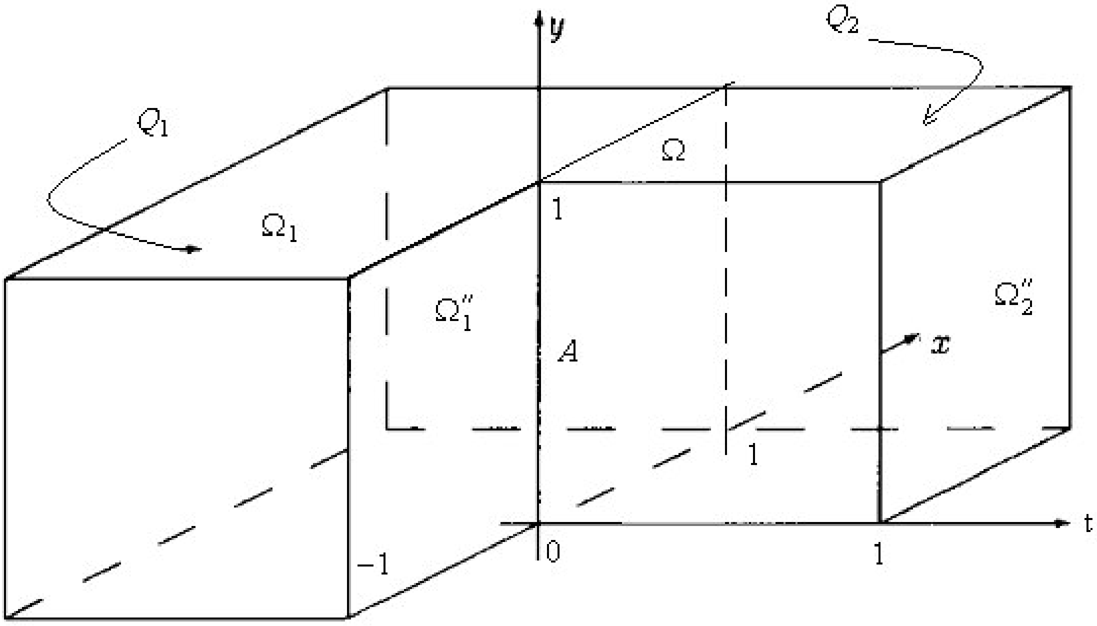

Fig. 8. The polyhedral domain Q.

5.2.2 Function spaces

We will need some anisotropic Sobolev spaces (see [43]), which we recall in the following definitions

$$H^{r,s}\left(\mathbb{R}^2\right) = \left\{u \in L^2\left(\mathbb{R}^2\right) : \left[\left(1+\zeta^2\right)^{r/2} + \left(1+\tau^2\right)^{s/2}\right]\widehat{u} \in L^2\left(\mathbb{R}^2\right)\right\}$$

where $\widehat{u}$ is the Fourier transform of u and r, s are two positive numbers. We put

$$H^{r,s}\left(\Omega\right) = \left\{u_{/\Omega} : u \in H^{r,s}\left(\mathbb{R}^2\right)\right\}, \tag{5.2.1}$$

with Ω is an open subset of $\mathbb{R}^2$. $H^{r,s}(\Omega)$ can also be defined as a real interpolation space between $H^{r/(1-\theta),s/(1-\theta)}(\Omega)$ and $L^2(\Omega)$, $\theta \in]0,1[$, (see [67])

$$H^{r,s}\left(\Omega\right) = \left[H^{r/(1-\theta),s/(1-\theta)}\left(\Omega\right), L^2\left(\Omega\right)\right]_{\theta}. \tag{5.2.2}$$

In this work, we consider the case $s = 2r$, $\theta = 1-r$,

$$H^{r,2r}\left(\Omega\right) = \left[H^{1,2}\left(\Omega\right), L^2\left(\Omega\right)\right]_{1-r} \quad \forall r \in]0,1[. \tag{5.2.3}$$

Putting $s = 2r$ in Relationship (5.2.1), we obtain

$$H^{r,2r}(\Omega) = \left\{u_{/\Omega} : u \in H^{r,2r}\left(\mathbb{R}^2\right)\right\}. \tag{5.2.4}$$

We have

$$\left\{u_{/\Omega} : u \in H^{r,2r}\left(\mathbb{R}^2\right)\right\} \subset \left[H^{1,2}(\Omega), L^2(\Omega)\right]_{1-r} \quad \forall r \in]0,1[,$$

and if Ω has "the horn property" of Besov [9], then

$$\left\{u_{/\Omega} : u \in H^{r,2r}\left(\mathbb{R}^2\right)\right\} = \left[H^{1,2}(\Omega), L^2(\Omega)\right]_{1-r} \quad \forall r \in]0,1[.$$

5.2.3 Properties of solutions to Problem (5.1.1) in the model domain Q

In Q, we consider the boundary value problem

$$\begin{cases} Lu = f \in L^2(Q) \\ u_{/\partial Q-\left(\Omega_1''\cup\Omega_2''\right)} = 0, \end{cases} \tag{5.2.5}$$

∂Q is the boundary of Q.

Throughout this section, f stands for an arbitrary fixed element of $L^2(Q)$ and $f_i = f_{/Q_i}$, $i = 1{,}2$. Recall the following result (see [43])

Theorem 5.2.1 *The problem*

$$\begin{cases} Lu_1 = f_1 \in L^2(Q_1) \\ u_{1/\partial Q_1-\left(\Omega\cup A\cup\Omega_1''\right)} = 0, \end{cases} \tag{5.2.6}$$

admits a (unique) solution $u_1 \in H^{1,2}(Q_1)$.

Hereafter, we denote the trace $u_{1/\Omega}$ by φ, which is in the Sobolev space $H^1(\Omega)$ because $u_1 \in H^{1,2}(Q_1)$ (cf. [19]).

Now, consider the following problem on Q_2

$$\begin{cases} Lu_2 = f_2 \in L^2(Q_2) \\ u_{2/\Omega} = \varphi \in H^1(\Omega), \\ u_{2/\partial Q_2-\left(\Omega\cup\Omega_2''\right)} = 0. \end{cases} \tag{5.2.7}$$

It is known that this problem admits a unique solution u_2 in $H^{1,2}(Q_2)$ if and only if some compatibility conditions are fulfilled, that is, $\varphi \in H_0^1(\Omega)$ i.e., (cf. [43])

$$\begin{cases} \text{a) } \varphi_{/\partial\Omega - A} & = & 0 \\ \text{b) } \varphi_{/A} & = & 0. \end{cases}$$

Remark 5.2.1 *We can observe that the boundary conditions of Problem (5.2.6) yield* $\varphi_{/\partial\Omega - A} = 0$*. So the compatibility condition a) is automatically satisfied. On the other hand, we have* $\varphi_{/A} = u_{1/A}$ *but we do not know whether* $u_{1/A}$ *vanishes. This is the reason why singularities may arise in the solution* u_2 *of Problem (5.2.7), and consequently, in the solution* u *of Problem (5.2.5).*

Remark 5.2.2 *The solution* u *of Problem (5.2.5) will be defined by*

$$u = \begin{cases} u_1 \text{ in } Q_1 \\ u_2 \text{ in } Q_2, \end{cases}$$

where u_1 *and* u_2 *are the solutions of (5.2.6) and (5.2.7) respectively. Observe that* $u_{1/\Omega} = u_{2/\Omega} = \varphi$ *and* $u_1 \in H^{1,2}(Q_1)$*. Therefore, if* $u_2 \in H^{1,2}(Q_2)$ *then* $u \in H^{1,2}(Q)$ *(cf. [19]). On the other hand,* u *is regular in* Q_1 *because* $u_{/Q_1} = u_1 \in H^{1,2}(Q_1)$*, and this means that the singularities which we seek are contained in* $u_{/Q_2}$*, i.e., in* u_2*. So, in the sequel, we will restrict ourselves to* u_2*.*

In the following result, we will introduce some functions $(P_j)_{j\in\mathbb{N}}$ which enable us to study the singular part of u_2.

Lemma 5.2.1 *Let*

$$\forall (x,y) \in \Omega,\ \forall j \in \mathbb{N},\ \ P_j(x,y) = \sin j\pi y \cos \frac{\pi}{2} x.$$

The functions $(P_j)_{j\in\mathbb{N}}$ *have the following properties*

a) $P_{j/\partial\Omega - A} = 0\ \forall j \in \mathbb{N}$,

b) $P_{j/A} = \sin j\pi y\ \forall j \in \mathbb{N},\ \forall y \in]0,1[$,

c) $P_j \in H^1(\Omega)\ \forall j \in \mathbb{N}$.

The determination of the smoothness of the singularities arising in u_2 needs the study of the following problem set in Q_2

$$\begin{cases} Lw_j = 0, \\ w_{j/\Omega} = P_j \ , \ \ \forall j \in \mathbb{N}, \\ w_{j/\partial Q_2 - \left(\Omega \cup \Omega_2''\right)} = 0, \end{cases} \tag{5.2.8}$$

where P_j are the functions defined in Lemma 5.2.1. Problem (5.2.8) admits a unique solution $w_j \in L^2(Q_2)$ for $j \in \mathbb{N}$ (see, for instance, [48]). So, we can define the function v on Ω by

$$v = u_2 - \sum_{j \in \mathbb{N}} a_j w_j$$

where u_2 is the solution of Problem (5.2.7) and $(a_j)_{j\in\mathbb{N}}$ are the coefficients of the decomposition of $\varphi_{/A}$ in $L^2(A)$

$$\varphi_{/A} = \sum_{j \in \mathbb{N}} a_j P_j.$$

Indeed, the functions $P_{j/A} = \sin j\pi y$ are an orthogonal basis of $L^2(A)$. So, by virtue of properties of $(P_j)_{j\in\mathbb{N}}$ in Lemma 5.2.1, it is easy to verify that

$$v_{/\Omega} = \varphi - \sum_{j \in \mathbb{N}} a_j P_j \in H_0^1(\Omega)$$

and v is the (unique) solution of the following problem

$$\begin{cases} Lv = f_2 \in L^2(Q_2), \\ v_{/\Omega} = \varphi - \sum_{j \in \mathbb{N}} a_j P_j, \\ v_{/\partial Q_2 - \left(\Omega \cup \Omega_2''\right)} = 0. \end{cases}$$

Therefore $v \in H^{1,2}(Q_2)$ and we have proved the following result

Proposition 5.2.1 *The solution u_2 of Problem (5.2.7) may be written as*

$$u_2 = v + \sum_{j \in \mathbb{N}} a_j w_j$$

where $v \in H^{1,2}(Q_2)$ and w_j stands for the solution of Problem (5.2.8), for $j \in \mathbb{N}$.

Remark 5.2.3 *Observe that v is the regular part of the solution u_2 while $\sum_{j\in\mathbb{N}} a_j w_j$ denotes the singular part which is generated by the singularities w_j, $j \in \mathbb{N}$. We also note that these singularities are solutions of Problem (5.2.8) and are independent of the data f in Problem (5.2.5). However, the singular part $\sum_{j\in\mathbb{N}} a_j w_j$ depends on f via the coefficients a_j, $j \in \mathbb{N}$. Indeed, a_j, $j \in \mathbb{N}$ depend on φ and the solution u_1 corresponding to the second member $f_1 = f_{/Q_1}$ is such that $u_{1/\Omega} = \varphi$.*

5.2.4 Smoothness of the singular solutions $(w_j)_{j\in\mathbb{N}}$

The sequence of functions $(\varphi_{j,k})_{j,k\in\mathbb{N}}$ defined by $\varphi_{j,k}(x,y) = 2\sin j\pi y \sin k\pi x$ in Ω is an orthonormal basis of $L^2(\Omega)$, formed by eigenfunctions of the operator $\Lambda = -\partial_x^2 - \partial_y^2$ with domain $D(\Lambda) = H^2(\Omega) \cap H_0^1(\Omega)$. Denote by $\lambda_{j,k} = 2\pi^2(j^2+k^2)$ the eigenvalue corresponding to each $\varphi_{j,k}$.

Since functions $(P_j)_{j\in\mathbb{N}}$ defined in Lemma 5.2.1 are in $L^2(\Omega)$, we may write

$$\forall i \in \mathbb{N} \quad P_i(x,y) = \sum_{j,k\in\mathbb{N}} b_{i,j,k}\varphi_{j,k}(x,y).$$

So, the solution w_i of Problem (5.2.8) can be written as

$$w_i(t,x,y) = \sum_{j,k\in\mathbb{N}} b_{i,j,k}\exp(-\lambda_{j,k}t)\,\varphi_{j,k}(x,y).$$

The particular form of the functions $\varphi_{j,k}$ and P_j enable us to write

$$P_j(x,y) = \sum_{k\in\mathbb{N}} b_{j,k}\varphi_{j,k}(x,y). \tag{5.2.9}$$

Integrating by parts in (5.2.9), we arrive at

$$b_{j,k} = \int_0^1 \sin k\pi y.\cos\frac{\pi}{2}y\,dy = \frac{4}{\pi}\frac{k}{1+4k^2}.$$

On the other hand, by (5.2.9), we deduce that the solution w_j of Problem (5.2.8) may be written as

$$w_j(t,x,y) = \sum_{k\in\mathbb{N}} b_{j,k}\exp(-\lambda_{j,k}t)\,\varphi_{j,k}(x,y). \tag{5.2.10}$$

Now, our aim is to determine the largest real number $r \in]0,1]$ such that $w_j \in H^{r,2r}(Q_2)$. This question will be treated in two steps. Recall that for $r \in]0,1]$

$$H^{r,2r}(Q_2) = H^r\left(]0,1[\,;L^2(\Omega)\right) \cap L^2\left(]0,1[\,;H^{2r}(\Omega)\right).$$

Step 1. When does w_j lie in $H^r(]0,1[\,;L^2(\Omega))$?

To answer this question we begin by extending w_j with respect to t and we set

$$\widetilde{w_j}(t,x,y) = \sum_{k\in\mathbb{N}} b_{j,k}\exp\left(-\lambda_{j,k}|t|\right)\varphi_{j,k}(x,y). \tag{5.2.11}$$

It is known that

$$\widetilde{w_j} \in H^r\left(\mathbb{R};L^2(\Omega)\right) \text{ if and only if } \left(1+t^2\right)^{r/2}\widehat{\widetilde{w_j}} \in L^2(\mathbb{R}\times\Omega)$$

where $\widehat{\widetilde{w_j}}$ denotes the Fourier transform of $\widetilde{w_j}$ with respect to t. So, (5.2.11) gives

$$\widehat{\widetilde{w_j}}(t,x,y) = \sum_{k\in\mathbb{N}} b_{j,k}\frac{\lambda_{j,k}}{\lambda_{j,k}^2+t^2}\varphi_{j,k}(x,y).$$

Therefore,

$$\begin{aligned}
\left\|(1+t^2)^{r/2}\widehat{\widetilde{w_j}}\right\|^2_{L^2(\mathbb{R}\times\Omega)} &= \int_{-\infty}^{+\infty}(1+t^2)^r\left\|\widehat{\widetilde{w_j}}\right\|^2_{L^2(\Omega)}dt \\
&= \int_{-\infty}^{+\infty}(1+t^2)^r\sum_{k\in\mathbb{N}} b_{j,k}^2\frac{\lambda_{j,k}^2}{(\lambda_{j,k}^2+t^2)^2}dt \\
&= \sum_{k\in\mathbb{N}} b_{j,k}^2\lambda_{j,k}^{2r-1}\int_{-\infty}^{+\infty}\frac{1}{\left(1+z^2\right)^{2-r}}dz.
\end{aligned}$$

We can observe that

$$\int_{-\infty}^{+\infty}\frac{1}{\left(1+z^2\right)^{2-r}}dz < \infty \Longleftrightarrow r < 3/2.$$

Then, if the condition $r < 3/2$ is satisfied,

$$\begin{aligned}
\sum_{k\in\mathbb{N}} b_{j,k}^2\lambda_{j,k}^{2r-1}\int_{-\infty}^{+\infty}\frac{1}{\left(1+z^2\right)^{2-r}}dz < \infty &\Longleftrightarrow \sum_{k\in\mathbb{N}} b_{j,k}^2\lambda_{j,k}^{2r-1} < \infty \\
&\Longleftrightarrow r < 3/4,
\end{aligned}$$

(note that the condition $r < 3/2$ is then satisfied). So the following result is proved

Proposition 5.2.2 *For all $j \in \mathbb{N}$, we have*

$$w_j \in H^r\left(]0,1[\,; L^2(\Omega)\right) \Longleftrightarrow r < 3/4.$$

Step 2. When does w_j lie in $L^2(]0,1[\,; H^{2r}(\Omega))$?

Using the fractional powers Λ^r of the operator $\Lambda = -\partial_x^2 - \partial_y^2$ (cf. [67]), we obtain, by (5.2.10),

$$\Lambda^r w_j = \sum_{k\in\mathbb{N}} b_{j,k} \lambda_{j,k}^r \exp(-\lambda_{j,k} t)\, \varphi_{j,k} \quad \text{for } r \in]0,1[\,. \tag{5.2.12}$$

On the other hand, we have

$$\|w_j\|^2_{L^2(]0,1[;H^{2r}(\Omega))} = \int_0^1 \|w_j(t,.,.)\|^2_{H^{2r}(\Omega)}\, dt. \tag{5.2.13}$$

But the domain $D(\Lambda^r)$ of the operator Λ^r may be obtained by interpolation (we use here the notations of [43])

$$D(\Lambda^r) = \left[D(\Lambda), L^2(\Omega)\right]_{1-r} \subset H^{2r}(\Omega)$$

with the equivalence of the norms of $D(\Lambda^r)$ and $H^{2r}(\Omega)$. Then we deduce, by (5.2.13), the equivalence of the two norms $\|w_j\|_{L^2(]0,1[;H^{2r}(\Omega))}$ and $\left(\int_0^1 \|w_j(t,.,.)\|^2_{D(\Lambda^r)}\, dt\right)^{1/2}$. So, relationship (5.2.12) shows that $w_j \in L^2(]0,1[\,; H^{2r}(\Omega))$ if and only if the series

$$\sum_{k\in\mathbb{N}} b_{j,k}^2 \lambda_{j,k}^{2r} \int_0^1 \exp(-2\lambda_{j,k} t)\, dt$$

is convergent. It is easy to see that this convergence holds if and only if $r < 3/4$.

Observe that this condition is the same as the necessary and sufficient condition obtained in Poposition 5.2.2. So, the following result is proved

Proposition 5.2.3

$$w_j \in L^2\left(]0,1[\,; H^{2r}(\Omega)\right) \Longleftrightarrow r < 3/4.$$

Now, our main result follows from Poposition 5.2.2 and Proposition 5.2.3, that is,

Theorem 5.2.2

$$w_j \in H^{r,2r}(Q_2) \Longleftrightarrow r < 3/4.$$

5.2.5 Smoothness of the singular part $\sum_{j\in\mathbb{N}} a_j w_j$

In order to study the behavior of the singular part $\sum_{j\in\mathbb{N}} a_j w_j$, one must know the behavior of the coefficients a_j, $j\in\mathbb{N}$. We will need the following result

Lemma 5.2.2 *We have*

$$\sum_{j\in\mathbb{N}} j.a_j^2 < \infty.$$

Proof. Since $u_{1/\Omega} \in H^1(\Omega)$, we deduce from the trace theory (see, for instance, [21]) that $\varphi_{/A} \in H_{00}^{1/2}(A)$ (the closure in $H^{1/2}$ of the space of C^∞ functions with compact support in A, see [43]). So,

$$\varphi_{/A} = \sum_{j\in\mathbb{N}} a_j . \sin j\pi y \in H_{00}^{1/2}(A)$$

which means that

$$\left\| \sum_{j\in\mathbb{N}} a_j \sqrt{j} . \sin j\pi y \right\|_{L^2(A)} < \infty$$

or equivalently

$$\sum_{j\in\mathbb{N}} j a_j^2 < \infty.$$

■

Setting

$$U = \sum_{j\in\mathbb{N}} a_j w_j = \sum_{j,k\in\mathbb{N}} a_j b_{j,k} \exp\left(-\lambda_{j,k} t\right) \varphi_{j,k}(x,y).$$

We look for the largest $r > 0$ such that $U \in H^{r,2r}(Q_2)$.

Proposition 5.2.4 1)

$$U \in H^r\left(]0,1[\,; L^2(\Omega)\right) \iff r < 3/4.$$

2)

$$U \in L^2\left(]0,1[\,; H^{2r}(\Omega)\right) \iff r < 3/4.$$

Proof. 1) By using a similar argument like that used in the proof of Proposition 5.2.2, we see that

$$U \in H^r\left(]0,1[\,; L^2(\Omega)\right) \Longleftrightarrow \sum_{j,k\in\mathbb{N}} a_j^2 \frac{1}{k^2}\left(j^2+k^2\right)^{2r-1} < \infty.$$

Due to Lemma 5.2.2, the condition $r < 3/4$ guarantees the convergence of the above series. On the other hand, this convergence leads to the convergence of the series $\sum_{k\in\mathbb{N}} \frac{1}{k^{4-4r}}$ when $r < 3/4$.

2) By using a similar argument like that used in the proof of Proposition 5.2.3, we obtain

$$\begin{aligned} U \in L^2(]0,1[\,; H^{2r}(\Omega)) &\Longleftrightarrow \int_0^1 \|U\|^2_{H^{2r}(\Omega)}\, dt < \infty \\ &\Longleftrightarrow \int_0^1 \left\| \sum_{j,k\in\mathbb{N}} a_j b_{j,k} \lambda_{j,k}^r \exp\left(-\lambda_{j,k} t\right) \varphi_{j,k} \right\|^2_{L^2(\Omega)} dt < \infty \\ &\Longleftrightarrow \frac{1}{2}\sum_{j,k\in\mathbb{N}} a_j^2 b_{j,k}^2 \lambda_{j,k}^{2r-1}\left(\exp\left(-2\lambda_{j,k}\right)-1\right) < \infty. \end{aligned}$$

It is easy to see that this convergence holds if and only if $r < 3/4$, because $b_{j,k} \sim 1/k$ for each $j \in \mathbb{N}$ ■

From Poposition 5.2.4, we have

Theorem 5.2.3

$$U \in H^{r,2r}(Q_2) \Longleftrightarrow r < 3/4.$$

Remark 5.2.4 *Since the solution u of Problem (5.2.5) is defined by*

$$u = \begin{cases} u_1 \ \text{in } Q_1 \\ u_2 \ \text{in } Q_2, \end{cases}$$

where $u_1 \in H^{1,2}(Q_1)$ and $u_2 = v + U$ with $v \in H^{1,2}(Q_2)$ and U is singular, then we can write in Q

$$u = U_R + U_S$$

with $U_R \in H^{1,2}(Q)$ and

$$U_S \in H^{r,2r}(Q) \Longleftrightarrow r < 3/4.$$

Remark 5.2.5 *If $\varphi_{/A} = 0$ i.e., $u_{1/\Omega} \in H_0^1(\Omega)$, then $a_j = 0$ for each $j \in \mathbb{N}$. Therefore $u \in H^{1,2}(Q)$ since the singular part $U = \sum_{j\in\mathbb{N}} a_j w_j$ vanishes.*

5.3 Second approach

5.3.1 Cylindrical case

In this subsection, Problem 2.2.3 of Chapter 2 will be revisited. Indeed, let Ω_0 an open bounded set of $\mathbb{R}^n$ with boundary Γ and Q_0 the cylinder $\mathbb{R}_+ \times \Omega_0$ with lateral boundary $\Sigma = \mathbb{R}_+ \times \Gamma$. We assume that Ω_0 is convex or of class C^2.

Consider in Q_0 the following boundary value problem

$$\begin{cases} \partial_t u - \Delta u = 0 \ \text{ in } \mathbb{R}_+ \times \Omega_0 \\ u = 0 \ \text{ on } \Sigma \\ u\left(0, x\right) = u_0\left(x\right), \, x \in \Omega_0. \end{cases} \tag{5.3.1}$$

Our interest is the regularity of the solution u of (5.3.1) in terms of the regularity of the initial data u_0. The cases where $u_0 \in H_0^1\left(\Omega_0\right)$ or $u_0 \in L^2\left(\Omega_0\right)$ are treated in Theorem 2.2.2 of Chapter 2. Here, we look for an intermediate regularity result similar to those given in Theorem 2.2.2 of Chapter 2. More precisely, we consider the case of an initial data between $H_0^1\left(\Omega_0\right)$ and $L^2\left(\Omega_0\right)$. So, in the sequel we will assume that $u_0 \in H_0^r\left(\Omega_0\right)$, $0 \leq r \leq 1$ where

$$H_0^r\left(\Omega_0\right) = \left\{u \in H^r\left(\Omega_0\right); u = 0 \text{ on } \Gamma\right\}$$

for $1/2 < r \leq 1$.

$$H_0^r\left(\Omega_0\right) = H_{00}^{1/2}\left(\Omega_0\right)$$

for $r = 1/2$,

$$H_0^r\left(\Omega_0\right) = H^r\left(\Omega_0\right)$$

for $0 \leq r < 1/2$.

Thus $H_0^r\left(\Omega_0\right)$ is the interpolation space $\left[H_0^1\left(\Omega_0\right), L^2\left(\Omega_0\right)\right]_{1-r}$ of order $1-r$ between $H_0^1\left(\Omega_0\right)$ and $L^2\left(\Omega_0\right)$. We look for the regularity of u in term of r. For $0 \leq r \leq 1$, we recall that the space $H^{r,2r}\left(Q_0\right)$ can be defined by

$$H^{r,2r}\left(Q_0\right) = L^2\left(\mathbb{R}_+; H^{2r}\left(\Omega_0\right)\right) \cap H^r\left(\mathbb{R}_+; L^2\left(\Omega_0\right)\right).$$

Lemma 5.3.1 *Let $u_0 \in L^2(\Omega_0)$. Then the solution u of Problem (5.3.1) associated to u_0 is in $H^{1/2}(\mathbb{R}_+; L^2(\Omega_0))$. Moreover, there exists a positive constant C (independent of u_0) such that*

$$\|u\|_{H^{1/2}(\mathbb{R}_+;L^2(\Omega_0))} \le C \|u_0\|_{L^2(\Omega_0)}.$$

Proof. Consider a sequence of spectral elements (λ_k, φ_k), $k \in \mathbb{N}$ of the Dirichlet problem for the Laplace operator

$$\begin{cases} -\Delta\varphi_k = \lambda_k \varphi_k \\ \varphi_k \in H_0^1(\Omega_0) \\ \|\varphi_k\|_{L^2(\Omega_0)} = 1. \end{cases}$$

The sequence $(\varphi_k)_{k\in\mathbb{N}}$ is a basis of $L^2(\Omega_0)$. If $u_0 \in L^2(\Omega_0)$ we may write

$$u_0(x) = \sum_{k\in\mathbb{N}} c_k \varphi_k(x)$$

with $\|u_0\|^2_{L^2(\Omega_0)} = \sum_{k\in\mathbb{N}} c_k^2$. The solution associated to u_0 is

$$u(t,x) = \sum_{k\in\mathbb{N}} c_k \exp(-\lambda_k t) \varphi_k(x).$$

Note $\widetilde{u}$ the extension of u to $\mathbb{R}$, i.e.,

$$\widetilde{u}(t,x) = \sum_{k\in\mathbb{N}} c_k \exp(-\lambda_k |t|) \varphi_k(x).$$

By the Fourier transform

$$\widehat{\widetilde{u}}(\zeta,x) = 2\sum_{k\in\mathbb{N}} \frac{c_k\lambda_k}{\zeta^2+\lambda_k^2}\varphi_k(x),$$

from which

$$\left\|\widehat{\widetilde{u}}(\zeta,.)\right\|^2_{L^2(\Omega_0)} = \sum_{k\in\mathbb{N}} 4c_k^2 \frac{\lambda_k^2}{(\zeta^2+\lambda_k^2)^2}$$

and by elementary calculations, we check easily that

$$\int_{\mathbb{R}} |\zeta| \left\|\widehat{\widetilde{u}}(\zeta,.)\right\|^2_{L^2(\Omega_0)} d\zeta = 4\pi \sum c_k^2.$$

Consequently

$$\widetilde{u} \in H^{1/2}\left(\mathbb{R}; L^2(\Omega_0)\right),$$

then

$$u \in H^{1/2}\left(\mathbb{R}_+; L^2(\Omega_0)\right)$$

by restriction of $\widetilde{u}$ to $t > 0$. ■

Our second main result in this work is

Theorem 5.3.1 *For given u_0 in $H_0^r(\Omega_0)$ $0 \leq r \leq 1$, Problem (5.3.1) has a unique weak solution in $H^{(1+r)/2,1+r}(Q_0)$.*

Proof. Let $u_0 \in H_0^r(\Omega_0)$ $0 \leq r \leq 1$, then $u_0 \in L^2(\Omega_0)$ and consequently (5.3.1) admits a unique weak solution (see Theorem 2.2.2) u in $L^2(\mathbb{R}_+; H_0^1(\Omega_0))$. In order to show that this solution is in $L^2(\mathbb{R}_+; H^{1+r}(\Omega_0))$ it suffices to interpolate the operator S which associates u to u_0. Indeed $S : u_0 \mapsto u$ is linear continuous from $H_0^1(\Omega_0)$ into $L^2(\mathbb{R}_+; H^2(\Omega_0))$ and from $L^2(\Omega_0)$ into $L^2(\mathbb{R}_+; H^1(\Omega_0))$ (see Theorem 2.2.2). By interpolation, it is linear continuous from $[H_0^1(\Omega_0), L^2(\Omega_0)]_{1-r}$ into $[L^2(\mathbb{R}_+; H^2(\Omega_0)), L^2(\mathbb{R}_+; H^1(\Omega_0))]_{1-r}$. But, thanks to some interpolation theory properties (see Triebel [67]):

$$\begin{aligned}
[L^2(\mathbb{R}_+; H^2(\Omega_0)), L^2(\mathbb{R}_+; H^1(\Omega_0))]_{1-r} &= L^2\left(\mathbb{R}_+; [H^2(\Omega_0), H^1(\Omega_0)]_{1-r}\right) \\
&= L^2(\mathbb{R}_+; H^{1+r}(\Omega_0)).
\end{aligned}$$

Then, S is linear continuous from $H_0^r(\Omega_0)$ into $L^2(\mathbb{R}_+; H^{1+r}(\Omega_0))$.

We can interpolate again S for proving $u \in H^{(1+r)/2}(\mathbb{R}_+; L^2(\Omega_0))$. Indeed, S is linear continuous from $L^2(\Omega_0)$ into $H^{1/2}(\mathbb{R}_+; L^2(\Omega_0))$ (see Lemma 5.3.1) and from $H_0^1(\Omega_0)$ into $H^1(\mathbb{R}_+; L^2(\Omega_0))$ (see Theorem 2.2.2). By interpolation, it is linear continuous from $[H_0^1(\Omega_0), L^2(\Omega_0)]_{1-r}$ into $\left[H^{1/2}(\mathbb{R}_+; L^2(\Omega_0)), H^1(\mathbb{R}_+; L^2(\Omega_0))\right]_{1-r}$. But, (see, Triebel [67])

$$\left[H^{1/2}\left(\mathbb{R}_+; L^2(\Omega_0)\right), H^1\left(\mathbb{R}_+; L^2(\Omega_0)\right)\right]_{1-r} = H^{(1+r)/2}\left(\mathbb{R}_+; L^2(\Omega_0)\right).$$

Then, S is linear continuous from $H_0^r(\Omega_0)$ into $H^{(1+r)/2}(\mathbb{R}_+; L^2(\Omega_0))$. This ends the proof of Theorem 5.3.1. ■

Remark 5.3.1 *The result of Theorem 5.3.1 is valid if $Q_0 =]0, T[\times \Omega_0$ with $T > 0$ instead of $Q_0 = \mathbb{R}_+ \times \Omega_0$.*

Corollary 5.3.1 *The problem*

$$\begin{cases} \partial_t v - \Delta v = f \in L^2(Q_0) \\ v = 0 \quad on\ \Sigma \\ v(0,x) = u_0(x) \in H_0^r(\Omega_0),\ 0 \le r \le 1, \end{cases} \tag{5.3.2}$$

admits a unique solution $v \in H^{(1+r)/2,1+r}(Q_0)$.

Proof. Let v_1 and v_2 be the solutions of the following problems

$$\begin{cases} \partial_t v_1 - \Delta v_1 = 0 \ \text{ in } Q_0 \\ v_1 = 0 \ \text{ on } \Sigma \\ v_1(0,x) = u_0(x) \in H_0^r(\Omega_0),\ 0 \le r \le 1, \end{cases} \tag{5.3.3}$$

$$\begin{cases} \partial_t v_2 - \Delta v_2 = f \in L^2(Q_0) \\ v_2 = 0 \ \text{ on } \Sigma \\ v_2(0,x) = 0. \end{cases} \tag{5.3.4}$$

It is well known that Problem (5.3.4) admits a unique solution $v_2 \in H^{1,2}(Q_0)$. Thanks to Theorem 5.3.1 we know that Problem (5.3.3) admits a unique solution $v_1 \in H^{(1+r)/2,1+r}(Q_0)$. Then, the linearity of the operator $\partial_t - \Delta$ ends the proof. ■

5.3.2 Non-cylindrical case

In this subsection, we combine results of Sections 5.2 and 5.3 to obtain new results of existence, uniqueness with optimal regularity for the heat equation in a domain which is the union of two cylinders of $\mathbb{R}^3$.

Let Ω_1, Ω_2 be two bounded open sets of $\mathbb{R}^2$ with boundaries Γ_1, Γ_2, respectively. We denote C_1, C_2 the cylinders $]0, T_1[\times\Omega_1$, $]T_1, T_2[\times\Omega_2$ (T_1, T_2 are finite positive numbers such that $T_1 < T_2$), with lateral boundaries $\Sigma_1 =]0, T_1[\times\Gamma_1$, $\Sigma_2 =]T_1, T_2[\times\Gamma_2$, respectively. We assume that Ω_1, Ω_2 are convex or of class C^2 and we denote the union of C_1, C_2 by C. So, we can distinguish four cases:

a) $\Omega_1 \cap \Omega_2 = \varnothing$: This case has no interest here because the solution u_1 in C_1 is independent of the solution u_2 in C_2; then the study is the same as in the previous subsection.

$b)$ $\Omega_1 \subseteq \Omega_2$, then $C = C_1 \cup C_2 \cup (\{T_1\} \times \Omega_1)$.

$c)$ $\Omega_2 \subsetneq \Omega_1$, then $C = C_1 \cup C_2 \cup (\{T_1\} \times \Omega_2)$.

$d)$ $\Omega_1 \cap \Omega_2 \neq \varnothing$, with $\Omega_1 \nsubseteq \Omega_2$ and $\Omega_2 \nsubseteq \Omega_1$: This case can be deduced from c).

So, it is sufficient to see the cases $b)$ and $c)$. In the sequel, f stands for an arbitrary fixed element of $L^2(C)$ and $f_i = f_{/C_i}$, $i = 1{,}2$. Our goal is to study the regularity of the solution of the heat equation in C when $\Omega_1 \subseteq \Omega_2$ or $\Omega_2 \subsetneq \Omega_1$.

The case where $\Omega_1 \subseteq \Omega_2$

Here $C = C_1 \cup C_2 \cup (\{T_1\} \times \Omega_1)$.

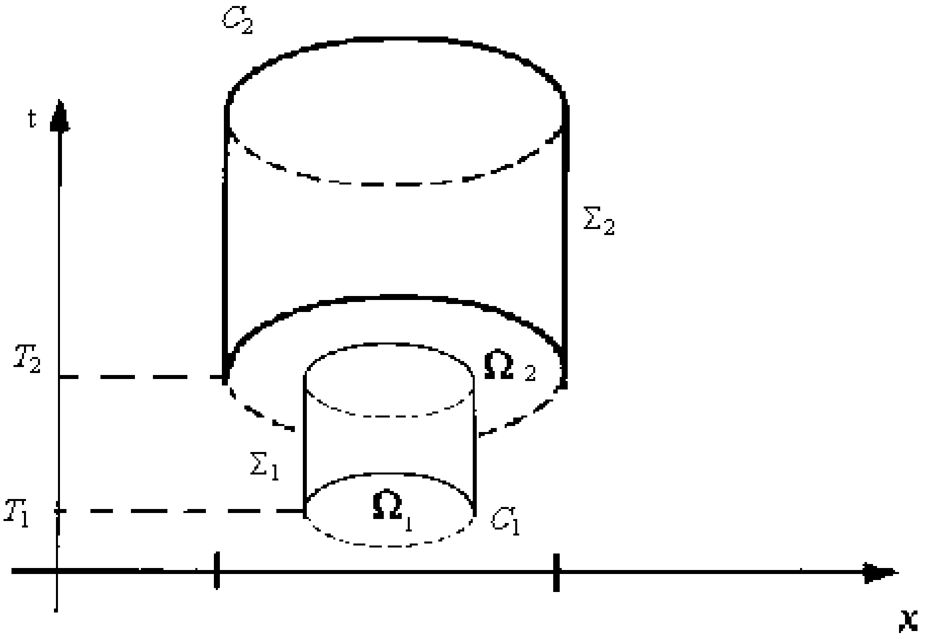

Fig 9 : The non-cylindrical domain $C = C_1 \cup C_2 \cup (\{T_1\} \times \Omega_1)$.

Consider in C, the following boundary value problem

$$\begin{cases} \partial_t u - \Delta u = f \in L^2(C) \\ u = 0 \quad \text{on } \Sigma \\ u(0, x) = 0,\ x \in \Omega_1 \end{cases} \tag{5.3.5}$$

where $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \sigma$ with $\sigma = \{T_1\} \times (\Omega_2 \backslash \Omega_1))$. The study is the same if we replace initial condition $u(0,x) = 0$ by $u(0,x) = u_0(x) \in H_0^1(\Omega_1)$ in (5.3.5). Recall the following result (see [43])

Lemma 5.3.2 *The Problem*

$$\begin{cases} \partial_t u_1 - \Delta u_1 = f_1 \in L^2(C_1) \\ u_{1/\partial C_1 - (\{T_1\} \times \Omega_1)} = 0, \end{cases} \tag{5.3.6}$$

admits a (unique) solution $u_1 \in H^{1,2}(C_1)$.

Hereafter, we denote the trace $u_{1/\{T_1\}\times\Omega_1}$ by ψ, which is in the Sobolev space $H_0^1(\{T_1\} \times \Omega_1)$ because $u_1 \in H^{1,2}(C_1)$ (cf. [20]). Let $\widetilde{\psi}$ be the 0-extension of ψ to $\{T_1\} \times \Omega_2$, which is in the Sobolev space $H_0^1(\{T_1\} \times \Omega_2)$, since $\Omega_1 \subseteq \Omega_2$.

Now, let $u_2 \in H^{1,2}(C_2)$ be the solution (see [43]) of the following problem in C_2

$$\begin{cases} \partial_t u_2 - \Delta u_2 = f_2 \in L^2(C_2) \\ u_{2/\{T_1\}\times\Omega_2} = \widetilde{\psi}, \\ u_2 = 0 \text{ on } \Sigma_2. \end{cases} \tag{5.3.7}$$

The solution u of Problem (5.3.5) will be defined by

$$u = \begin{cases} u_1 \text{ in } C_1, \\ u_2 \text{ in } C_2. \end{cases}$$

It is no difficult to prove that $u \in H^{1,2}(C)$. So, the regularity of the solution of Problem (5.3.5) is optimal in this kind of non-cylindrical domain.

The case where $\Omega_2 \subsetneq \Omega_1$

Here $C = C_1 \cup C_2 \cup (\{T_1\} \times \Omega_2)$.

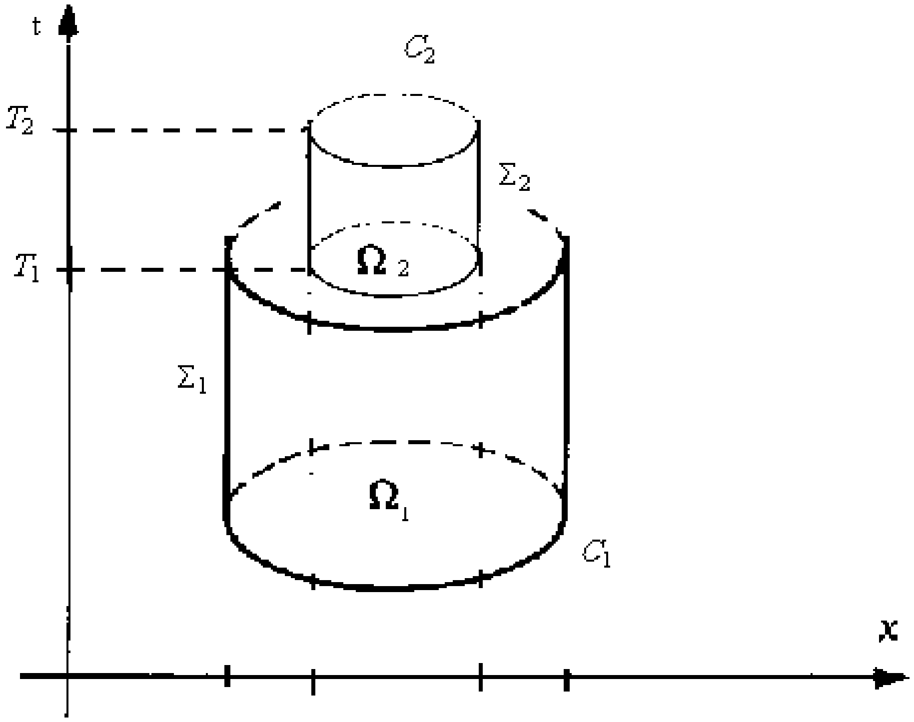

Fig 10 : The non-cylindrical domain $C = C_1 \cup C_2 \cup (\{T_1\} \times \Omega_2)$.

Consider in C, the following boundary value problem

$$\begin{cases} \partial_t v - \Delta v = f \in L^2(C) \\ v = 0 \quad \text{on } \Sigma_1 \cup \Sigma_2 \\ v(0, x) = 0, \; x \in \Omega_1 \end{cases} \tag{5.3.8}$$

It is well known (see Lemma 5.3.2) that Problem (5.3.8) in C_1 admits a unique solution $v_1 \in H^{1,2}(C_1)$. Hereafter, we denote the trace $v_{1/\{T_1\}\times\Omega_1}$ by φ, which is in the Sobolev space $H_0^1(\{T_1\} \times \Omega_1)$. So, $v_{1/\{T_1\}\times\Omega_2} \in H^r(\{T_1\} \times \Omega_2) = H_0^r(\{T_1\} \times \Omega_2)$ for $0 \le r < \frac{1}{2}$.

Now, consider the following problem in C_2

$$\begin{cases} \partial_t v_2 - \Delta v_2 = f_2 \in L^2(C_2) \\ v_{2/\{T_1\}\times\Omega_2} = \varphi, \\ v_2 = 0 \text{ on } \Sigma_2. \end{cases} \tag{5.3.9}$$

Thanks to Theorem 5.3.1 we have

Lemma 5.3.3 *Problem (5.3.9) admits a unique solution* $u_2 \in H^{(1+r)/2,1+r}(C_2)$.

The main result of this section follows from Lemma 5.3.2, Lemma 5.3.3 and Theorem 5.2.3, that is,

Theorem 5.3.2 *For each* $f \in L^2(C)$, *the (unique) solution* u *of Problem (5.3.8) is such that*

1) $u_{/C_1} \in H^{1,2}(C_1)$.

2) $u_{/C_2} \in H^{(1+r)/2,1+r}(C_2)$ *if and only if* $r < 1/2$,

or equivalently,

$u_{/C_2} \in H^{r,2r}(C_2)$ *if and only if* $r < 3/4$.

Remark 5.3.2 *The result of Theorem 5.3.2 is valid if* C_1, C_2 *are cylinders of* $\mathbb{R}^n$ $(n > 3)$.

Remark 5.3.3 *Using similar arguments in the case where* $\Omega_1 \cap \Omega_2 \neq \varnothing$, *with* $\Omega_1 \nsubseteqq \Omega_2$ *and* $\Omega_2 \nsubseteqq \Omega_1$, *we can obtain a result similar to Theorem 5.3.2.*