CHAPTER 4

Parabolic equations with Robin type boundary conditions in a non-rectangular domain

Abstract. We are concerned in this work, with the parabolic equation

$$\partial_t u - c^2(t)\,\partial_x^2 u = f \in L^2(\Omega)$$

subject to Robin type boundary conditions and set in the non-rectangular domain

$$\Omega = \left\{(t,x) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x < \varphi_2(t)\right\}.$$

Our aim is to find some conditions on coefficient c and functions $(\varphi_i)_{i=1,2}$ such that the solution of this equation belongs to the anisotropic Sobolev space

$$H^{1,2}(\Omega) = \left\{u \in L^2(\Omega) : \partial_t u, \partial_x u, \partial_x^2 u \in L^2(\Omega)\right\}.$$

Key words. Parabolic equations, non-rectangular domains, Robin condition, anisotropic Sobolev spaces.

4.1 Introduction

Let Ω be an open set of $\mathbb{R}^2$ defined by

$$\Omega = \left\{ (t,x) \in \mathbb{R}^2 : 0 < t < T,\ \varphi_1(t) < x < \varphi_2(t) \right\}$$

where T is a finite positive number, while φ_1 and φ_2 are continuous real-valued functions defined on $[0,T]$, Lipschitz continuous on $]0,T]$, and such that

$$\varphi_1(t) < \varphi_2(t)$$

for $t \in]0,T]$. The lateral boundary of Ω is defined by

$$\Gamma_i = \left\{ (t, \varphi_i(t)) \in \mathbb{R}^2 : 0 < t < T \right\},\ i = 1,\ 2.$$

Set

$$\varphi_2 - \varphi_1 = \varphi.$$

We will then assume that

$$\varphi(0) = 0, \tag{4.1.1}$$

$$\varphi_i'(t)\,\varphi(t) \quad \rightarrow \quad 0 \qquad \text{as } t \rightarrow 0, \quad i = 1,\ 2. \tag{4.1.2}$$

In Ω, we consider the boundary value problem

$$\begin{cases} \partial_t u - c^2(t)\,\partial_x^2 u = f \in L^2(\Omega) \\ \\ b_i(t)\,\partial_x u + a_i(t)\,u_{/\Gamma_i} = 0,\ i = 1,\ 2, \end{cases} \tag{4.1.3}$$

where coefficients $a_i(t)$ and $b_i(t)$ are such that

$$(a_i(t), b_i(t)) \neq (0,0),\ i = 1{,}2, \tag{4.1.4}$$

for every $t \in]0,T[$, which correspond to Robin type boundary conditions.

The case

$$a_i(t) \neq 0 \text{ and } b_i(t) = 0,\ i = 1{,}2,$$

for every $t \in]0,T[$, corresponds to Dirichlet boundary conditions. We can find in [31], [32], [61] and [62] solvability results of this kind of problems. In Sadallah [62], the same equation is studied by another approach making use of the so-called Schur's Lemma and gives the same result obtained in [61] by the *a priori* estimates technique. In [31] and [32], the authors deal with the heat equation (i.e., the case where $c(t) = 1$) set in a non-rectangular domain with a right-hand side taken in L^p, where $p \in]1, \infty[$, and have obtained optimal regularity results by the operators sum method. These results are generalized in [33] to a parabolic equation of the type

$$\partial_t u(t,x) - \partial_x^2 u(t,x) + \lambda m(t,x)\, u(t,x) = f(t,x)$$

where λ is a positive spectral parameter and $m(.)$ some positive weight functions. Regularity results of the heat equation solution in two-space dimensionnal case are obtained in [26] by using the domain decomposition method.

The case

$$a_i(t) = 0 \text{ and } b_i(t) \neq 0,\ i = 1{,}2,$$

for every $t \in]0, T[$, corresponds to Neumann type boundary conditions. Hofmann and Lewis [22] have also considered the classical heat equation with Neumann boundary condition in non-cylindrical domains satisfying some conditions of Lipschitz's type. The authors showed that the optimal L^p regularity holds for $p = 2$ and the situation gets progressively worse as p approaches 1.

The two boundary conditions on each lateral boundary Γ_i, $i = 1, 2$ of Ω may be of different type. We can find in [64] an abstract study for parabolic problems with mixed (Dirichlet-Neumann) lateral boundary conditions in the Hilbertian case. The author obtains some regularity results under assumption on the geometrical behavior of the boundary which cannot include our triangular domain.

Thanks to (4.1.4), Problem (4.1.3) may be written in the form

$$\begin{cases} \partial_t u - c^2(t)\, \partial_x^2 u = f \in L^2(\Omega) \\ \\ \partial_x u + \beta_i(t)\, u_{/\Gamma_i} = 0,\ i = 1,\ 2, \end{cases} \tag{4.1.5}$$

where

$$\beta_i(t) = a_i(t) / b_i(t)$$

for every $t \in]0,T[$. Here, c is a bounded differentiable coefficient depending on time such that

$$0 < d_1 \leq c(t) \leq d_2 \tag{4.1.6}$$

$$0 < m_1 \leq c(t)\, c'(t) \leq m_2 \tag{4.1.7}$$

for every $t \in]0,T[$, where d_1, d_2, m_1 and m_2 are constants.

The coefficients $(\beta_i)_{i=1,2}$ are continuous real-valued functions on $]0,T[$ such that

$$\beta_1(t) < 0 \text{ and } \beta_2(t) > 0 \text{ for all } t \in [0,T]. \tag{4.1.8}$$

We also assume that

$$\beta_1 c^2 \text{ is an increasing function on }]0,T[, \tag{4.1.9}$$

$$\beta_2 c^2 \text{ is a decreasing function on }]0,T[, \tag{4.1.10}$$

$$\left| \frac{1+\beta_2(t)}{A(t)} \right| \leq l \tag{4.1.11}$$

and

$$\left| \frac{\beta_1(t)\,(1+\beta_2(t))}{A(t)} \right| \leq l, \tag{4.1.12}$$

for every $t \in]0,T[$, where $A(t) = \beta_1(t)\,\beta_2(t) + \beta_1(t) - \beta_2(t)$ and l is a positive constant. Note that hypothesis (4.1.8) on $(\beta_i)_{i=1,2}$ implies that $A(t) \neq 0$ for every $t \in]0,T[$.

A natural assumption between coefficients $(\beta_i)_{i=1,2}$ and functions of parametrization $(\varphi_i)_{i=1,2}$ of the domain Ω which guarantees the uniqueness of the solution of Problem (4.1.5) is

$$(-1)^i \left(c^2(t)\, \beta_i(t) - \frac{\varphi_i'(t)}{2} \right) \geq 0 \quad a.e.\ t \in]0,T[,\ i = 1, 2. \tag{4.1.13}$$

In this work, we will prove that Problem (4.1.5) has a solution with optimal regularity, that is a solution u belonging to the anisotropic Sobolev space

$$H_\gamma^{1,2}(\Omega) := \left\{ u \in H^{1,2}(\Omega) : \partial_x u + \beta_i(t)\, u_{/\Gamma_i} = 0,\ i = 1,2 \right\}$$

with

$$H^{1,2}(\Omega) = \left\{ u \in L^2(\Omega) : \partial_t u, \partial_x u, \partial_x^2 u \in L^2(\Omega) \right\}.$$

The most interesting point of the parabolic problem studied here is the fact that $\varphi(0) = 0$, which yields the domain Ω not rectangular. In this case, the domain Ω cannot be transformed into regular domains without the appearance of some degenerate terms in the parabolic equation (see, for example, Sadallah [58]).

The organization of this chapter is as follows. In Section 4.2, first we prove an uniqueness result for Problem (4.1.5), then we derive some technical lemmas which will allow us to prove an uniform estimate (in a sense to be defined later). In Section 4.3, there are two main steps. First, we prove that Problem (4.1.5) admits a (unique) solution in the case of a domain which can be transformed into a rectangle. Secondly, for T small enough, we prove that the result holds true in the case of a triangular domain under the above mentionned assumptions on coefficient c and functions $(\beta_i, \varphi_i)_{i=1,2}$. The method used here is based on the approximation of the triangular domain by a sequence of subdomains $(\Omega_n)_n$ which can be transformed into regular domains (rectangles) and we establish an uniform estimate of the type

$$\|u_n\|_{H^{1,2}(\Omega_n)} \leq K \|f\|_{L^2(\Omega_n)},$$

where u_n is the solution of the problem (4.1.5) in Ω_n and K is a constant independent of n, which allows us to pass to the limit. Finally, in Section 4.4 we show that the obtained local in time result can be extended to a global in time one.

4.2 Preliminaries

Proposition 4.2.1 *Under assumption (4.1.13), Problem (4.1.5) is uniquely solvable.*

Proof. Let us consider $u \in H^{1,2}_{\gamma}(\Omega)$ a solution of the problem (4.1.5) with a null right-hand side term. So,

$$\partial_t u - c^2(t)\, \partial_x^2 u = 0 \text{ in } \Omega.$$

In addition u fulfils the boundary conditions

$$\partial_x u + \beta_i(t)\, u_{/\Gamma_i} = 0,\ i = 1,\ 2.$$

Using Green formula, we have

$$\int_{\Omega} (\partial_t u - c^2(t)\,\partial_x^2 u)\, u \; dt \; dx = \int_{\partial\Omega} \left(\tfrac{1}{2}|u|^2 \nu_t - c^2(t)\, u \; \partial_x u \; \nu_x\right) d\sigma + \int_{\Omega} c^2(t)\,(\partial_x u)^2 \, dt \; dx$$

where ν_t, ν_x are the components of the unit outward normal vector at $\partial\Omega$. We shall rewrite the boundary integral making use of the boundary conditions. We obtain

$$\int_{\partial\Omega} \left(\tfrac{1}{2}|u|^2 \nu_t - c^2(t)\, u \; \partial_x u \; \nu_x\right) d\sigma = \sum_{i=1}^{2} \int_{\Gamma_i} (-1)^i \left(c^2(t)\,\beta_i(t) - \frac{\varphi_i'(t)}{2}\right) u^2(t, \varphi_i(t))\, dt + \frac{1}{2}\int_{\Gamma_3} u^2 dx$$

where $\Gamma_3 = \{(T, x) : \varphi_1(T) < x < \varphi_2(T)\}$. Consequently

$$\int_{\Omega} \left(\partial_t u - c^2(t)\,\partial_x^2 u\right) u \; dt \; dx = 0$$

yields the inequality

$$\int_{\Omega} c^2 . (\partial_x u)^2 \, dt \; dx \leq 0,$$

because

$$\sum_{i=1}^{2} \int_{\Gamma_i} (-1)^i \left(c^2(t)\,\beta_i(t) - \frac{\varphi_i'(t)}{2}\right) u^2(t, \varphi_i(t))\, dt + \frac{1}{2}\int_{\Gamma_3} u^2 dx \geq 0,$$

thanks to hypothesis (4.1.13). This implies that $\partial_x u = 0$ and consequently $\partial_x^2 u = 0$. Then, the equation of (4.1.5) gives $\partial_t u = 0$. Thus, u is constant. The boundary conditions and the fact that $\beta_i(t) \neq 0$ for all $t \in \left]0, T\right[$ imply that $u = 0$. ■

Lemma 4.2.1 *We assume that β_1 and β_2 fulfil the conditions (4.1.8), (4.1.11) and (4.1.12). Then, for a fixed $t \in \left]0, 1\right[$, there exists a positive constant K_1 independent of t, such that for each $u \in H_\gamma^2(0, 1)$*

$$\left\| u^{(j)} \right\|_{L^2(0,1)} \leq K_1 \left\| u^{(2)} \right\|_{L^2(0,1)}, \; j = 0,\, 1,$$

where

$$H_\gamma^2(0,1) = \left\{ u \in H^2(0,1) : u'(0) + \beta_1(t)\, u(0) = 0, \; u'(1) + \beta_2(t)\, u(1) = 0 \right\}.$$

Proof. Let $t \in]0,1[$ and f an arbitrary fixed element of $L^2(0,1)$. Every solution of the ordinary differential equation $u'' = f$ is of the form

$$u(y) = \int_0^y \left\{ \int_0^x f(s)\,ds \right\} dx + yu'(0) + u(0),\ y \in [0,1].$$

The variables $u(0)$ and $u'(0)$ are to be determined in a unique way such that the boundary conditions $u'(0) + \beta_1(t)\,u(0) = 0$ and $u'(1) + \beta_2(t)\,u(1) = 0$ are satisfied.

From the preceding representation of the solution (and thus also its derivative) and from the required Robin boundary conditions we obtain the following system to be solved:

$$\begin{cases} (1+\beta_2(t))\,u'(0) + \beta_2(t)\,u(0) = -\displaystyle\int_0^1 f(s)\,ds - \beta_2(t)\int_0^1 \left\{ \int_0^x f(s)\,ds \right\} dx \\ u'(0) + \beta_1(t)\,u(0) = 0 \end{cases}$$

This system in the unknowns $u(0)$ and $u'(0)$ is uniquely solvable if and only if

$$A(t) = \beta_1(t)\,\beta_2(t) + \beta_1(t) - \beta_2(t) \neq 0$$

for every $t \in]0,T[$. This condition is verified thanks to (4.1.8).

Finally, the unique solution of the problem

$$\begin{cases} u'' = f \\ u'(0) + \beta_1(t)\,u(0) = 0, \\ u'(1) + \beta_2(t)\,u(1) = 0, \end{cases}$$

is given by

$$u(y) = \int_0^y \left\{ \int_0^x f(s)\,ds \right\} dx + yu'(0) + u(0),$$

where

$$\begin{cases} u(0) = \dfrac{\displaystyle\int_0^1 f(s)\,ds + \beta_2(t)\int_0^1 \left\{ \int_0^x f(s)\,ds \right\} dx}{A(t)} \\ u'(0) = -\beta_1(t)\,u(0). \end{cases}$$

Using the Cauchy-Schwarz inequality, we obtain the following estimates

$$|u(0)| \leq C \left| \frac{(1+\beta_2(t))}{A(t)} \right| \|f\|_{L^2(0,1)}$$
$$|u'(0)| \leq C \left| \frac{\beta_1(t)\,(1+\beta_2(t))}{A(t)} \right| \|f\|_{L^2(0,1)},$$

which will allow us to obtain the desired estimates, thanks to the conditions (4.1.11) and (4.1.12). ■

Lemma 4.2.2 *Under the assumptions (4.1.8), (4.1.11) and (4.1.12) on* $(\beta_i)_{i=1,2}$ *and for a fixed* $t \in]0,1[$, *there exists a constant* C_1 *(independent of* a *and* b*) such that*

$$\left\|v^{(j)}\right\|^2_{L^2(a,b)} \leq C_1 (b-a)^{2(2-j)} \left\|v^{(2)}\right\|^2_{L^2(a,b)}, \ j = 0,\ 1,$$

for each $v \in H^2_\gamma(a,b)$, *with*

$$H^2_\gamma(a,b) = \left\{ v \in H^2(a,b) : v'(a) + \frac{\beta_1(t)}{b-a} v(a) = 0,\ v'(b) + \frac{\beta_2(t)}{b-a} v(b) = 0 \right\}.$$

Proof. It is a direct consequence of Lemma 4.2.1. Indeed, we define the following affine change of variable

$$\begin{array}{lcl} [0,1] & \rightarrow & [a,b] \\ x & \rightarrow & (1-x)a + xb = y \end{array}$$

and we set

$$u(x) = v(y).$$

Then if $u \in H^2_\gamma(0,1)$, v belongs to $H^2_\gamma(a,b)$. We have

$$\begin{aligned} \|u'\|^2_{L^2(0,1)} &= \int_0^1 (u')^2(x)\ dx \\ &= \int_a^b (v')^2(y)(b-a)^2\ \frac{dy}{b-a} \\ &= \int_a^b (v')^2(y)(b-a)\ dy \\ &= (b-a)\|v'\|^2_{L^2(a,b)}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|u''\|^2_{L^2(0,1)} &= \int_0^1 (u'')^2(x)\ dx \\ &= \int_a^b (v'')^2(y)(b-a)^3\ dy \\ &= (b-a)^3\|v''\|^2_{L^2(a,b)}. \end{aligned}$$

Using the inequality

$$\|u'\|^2_{L^2(0,1)} \leq K_1^2 \|u''\|^2_{L^2(0,1)}$$

of Lemma 4.2.1, we obtain the desired inequality

$$\|v'\|^2_{L^2(a,b)} \leq C_1 (b-a)^2 \|v''\|^2_{L^2(a,b)}$$

with $C_1 = K_1^2$.

The inequality

$$\|v\|_{L^2(a,b)}^2 \leq C_1 (b-a)^4 \|v''\|_{L^2(a,b)}^2$$

can be obtained by a similar method. ■

4.3 Local in time result

4.3.1 Case of a domain which can be transformed into a rectangle

In this subsection, we consider the case of a truncated domain Ω. Let

$$\Omega = \left\{(t,x) \in \mathbb{R}^2 : 0 < t < T,\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

where φ_1 and φ_2 are such that

$$\varphi(t) := \varphi_2(t) - \varphi_1(t) > 0$$

for all $t \in [0,T]$.

Theorem 4.3.1 *Under the assumptions (4.1.6), (4.1.8), (4.1.11), (4.1.12) and (4.1.13) on the coefficients $(\beta_i, c)_{i=1,2}$, the problem*

$$\begin{cases} \partial_t u - c^2(t)\, \partial_x^2 u = f \in L^2(\Omega), \\ u_{/t=0} = 0, \\ \partial_x u + \beta_i(t)\, u_{/x=\varphi_i(t)} = 0,\ i = 1,\ 2, \end{cases} \tag{4.3.1}$$

admits a (unique) solution $u \in H^{1,2}(\Omega)$.

Proof. The uniqueness of the solution is easy to check, thanks to (4.1.13). Let us prove the existence. The change of variable

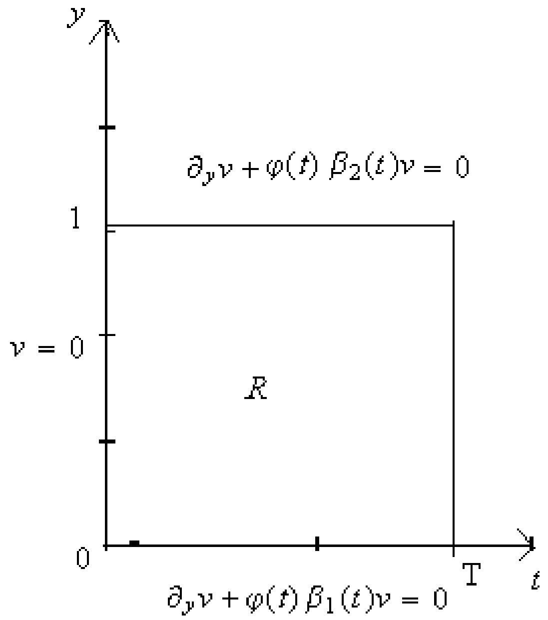
$$(t,x) \mapsto (t,y) = \left(t, \frac{x - \varphi_1(t)}{\varphi(t)}\right)$$

transforms Ω into the rectangle $R =]0, T[\times]0, 1[$. Putting $u(t,x) = v(t,y)$ and $f(t,x) = g(t,y)$, then Problem (4.3.1) becomes

$$\begin{cases} \partial_t v(t,y) + a(t,y)\,\partial_y v(t,y) - \dfrac{1}{b^2(t)}\partial_y^2 v(t,y) = g(t,y) \\ v_{/t=0} = 0, \\ \partial_y v + \varphi(t)\,\beta_1(t)\, v_{/y=0} = 0, \\ \partial_y v + \varphi(t)\,\beta_2(t)\, v_{/y=1} = 0, \end{cases} \tag{4.3.2}$$

where

$$b(t) = \frac{\varphi(t)}{c(t)}$$
$$a(t,y) = -\frac{y\varphi'(t) + \varphi_1'(t)}{\varphi(t)}.$$



$R =]0, T[\times]0, 1[.$

The change of variable defined above conserves the spaces $H^{1,2}$ and L^2. In other words

$$f \in L^2(\Omega) \Leftrightarrow g \in L^2(R)$$
$$u \in H^{1,2}(\Omega) \Leftrightarrow v \in H^{1,2}(R).$$

We need the following lemma:

Lemma 4.3.1 *The operator*

$$\begin{array}{rlll} B: & H_\gamma^{1,2}(R) & \rightarrow & L^2(R) \\ & v & \mapsto & Bv = a(t,y)\,\partial_y v \end{array}$$

is compact, where for a fixed $t \in \,]0,T[$

$$H_\gamma^{1,2}(R) = \left\{ v \in H^{1,2}(R) : v_{/\Gamma_0} = 0, \partial_y v + \varphi(t)\,\beta_i(t)\,v_{/\Gamma_{i,R}} = 0, i = 1,2 \right\},$$

with $\Gamma_0 = \{0\} \times]0,1[$, $\Gamma_{1,R} =]0,T[\times \{0\}$ *and* $\Gamma_{2,R} =]0,T[\times \{1\}$.

Proof. R has the "horn property" of Besov [9], so

$$\begin{array}{rlll} \partial_y: & H_\gamma^{1,2}(R) & \rightarrow & H^{\frac{1}{2},1}(R) \\ & v & \mapsto & \partial_y v \end{array}$$

is continuous. Since R is bounded, the canonical injection is compact from $H^{\frac{1}{2},1}(R)$ into $L^2(R)$, see for instance [9]. Here

$$H^{\frac{1}{2},1}(R) = L^2\left(0,T; H^1\,]0,1[\right) \cap H^{\frac{1}{2}}\left(0,T; L^2\,]0,1[\right),$$

see [43] for the complete definitions of the $H^{r,s}$ Hilbertian Sobolev spaces. Then, ∂_y is a compact operator from $H_\gamma^{1,2}(R)$ into $L^2(R)$. Since $a(.,.)$ is a bounded function, the operator $B = a\partial_y$ is also compact from $H_\gamma^{1,2}(R)$ into $L^2(R)$. ■

So, it is sufficient to show that the operator

$$\partial_t - \frac{c^2}{\varphi^2}\partial_y^2 : \; H_\gamma^{1,2}(R) \; \rightarrow \; L^2(R)$$

is an isomorphism. A simple change of variable $t = h(s)$ with $h'(s) = \frac{\varphi^2}{c^2}(t)$, transforms the problem

$$\begin{cases} \partial_t v(t,y) - \dfrac{c^2}{\varphi^2}(t)\,\partial_y^2 v(t,y) = g(t,y) \in L^2(R), \\ v_{/t=0} = 0, \\ \partial_y v + \varphi(t)\,\beta_1(t)\,v_{/y=0} = 0, \\ \partial_y v + \varphi(t)\,\beta_2(t)\,v_{/y=1} = 0, \end{cases}$$

into the following

$$\begin{cases} \partial_s w(s,y) - \partial_y^2 w(s,y) = \zeta(s,y) \\ w_{/s=h^{-1}(0)} = 0, \\ \partial_y w + \varphi\beta_1(h(s))\, w_{/y=0} = 0, \\ \partial_y w + \varphi\beta_2(h(s))\, w_{/y=1} = 0, \end{cases} \tag{4.3.3}$$

with $\zeta(s,y) = \dfrac{g(t,y)}{h'(s)}$ and $w(s,y) = v(t,y)$. Note that this change of variable preserves the spaces L^2 and $H^{1,2}$. It follows from Lions and Magenes [43], for instance, that there exists a unique $w \in H^{1,2}$ solution of the problem (4.3.3). This implies that Problem (4.3.1) admits a unique solution $u \in H^{1,2}(\Omega)$. We obtain the function u by setting $u(t,x) = v(t,y) = w(h^{-1}(t), y)$. This ends the proof of Theorem 4.3.1.

Lemma 4.3.2 *The space*

$$W = \left\{ u \in D\left([0,T]; H^2(0,1)\right) : \partial_x u + \beta_i(t)\, u_{/\Gamma_i} = 0,\ i = 1,2 \right\}$$

is dense in

$$H_\gamma^{1,2}(]0,T[\times]0,1[) = \left\{ u \in H^{1,2}(]0,T[\times]0,1[) : \partial_x u + \beta_i(t)\, u_{/\Gamma_i} = 0,\ i = 1,2 \right\}$$

where $\Gamma_1 =]0,T[\times\{0\}$ *and* $\Gamma_2 =]0,T[\times\{1\}$.

The above lemma is a particular case of [43, Vol.2, Theorem 2.1].

We shall need the following result in order to justify the calculus of the next section.

Lemma 4.3.3 *The space*

$$\left\{ u \in H^2(R); u_{/\Gamma_0} = 0,\ \partial_x u + \beta_i(t)\, u_{/\Gamma_i} = 0,\ i = 1,2 \right\}$$

is dense in the space

$$\left\{ u \in H^{1,2}(R);\ u_{/\Gamma_0} = 0,\ \partial_x u + \beta_i(t)\, u_{/\Gamma_i} = 0,\ i = 1,2 \right\},$$

where $\Gamma_0 = \{0\}\times]0,1[$, $\Gamma_1 =]0,T[\times\{0\}$, $\Gamma_2 =]0,T[\times\{1\}$ *and* $R =]0,T[\times]0,1[$.

Proof. It is a consequence of [43, Vol. 1, Theorem 2.1]. ■

Remark 4.3.1 *We can replace in Lemma 4.3.3, R by Ω with the help of the change of variable defined above.*

4.3.2 Case of a triangular domain

In this case, we define Ω by

$$\Omega = \left\{(t,x) \in \mathbb{R}^2 : 0 < t < T,\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

where φ_1 and φ_2 are such that

$$\varphi(t) := \varphi_2(t) - \varphi_1(t) > 0$$

for all $t \in]0,T]$ and

$$\varphi(0) := \varphi_2(0) - \varphi_1(0) = 0.$$

For each $n \in \mathbb{N}$, we define Ω_n by

$$\Omega_n = \left\{(t,x) \in \mathbb{R}^2 : a_n < t < T,\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

where $(a_n)_n$ is a decreasing sequence to zero. Thus, we have

$$\varphi(a_n) > 0.$$

Setting $f_n = f_{/\Omega_n}$, where $f \in L^2(\Omega)$. We denote $u_n \in H^{1,2}(\Omega_n)$ the solution of Problem (4.3.1) in Ω_n

$$\begin{cases} \partial_t u_n - c^2(t)\,\partial_x^2 u_n = f_n \in L^2(\Omega_n) \\ u_{n/t=a_n} = 0, \\ \partial_x u_n + \beta_i(t)\, u_{n/\Gamma_{n,i}} = 0,\ i = 1,\ 2, \end{cases}$$

where

$$\Gamma_{n,i} = \left\{(t, \varphi_i(t)),\ a_n < t < T\right\}, i = 1, 2.$$

Such a solution u_n exists by Theorem 4.3.1.

Theorem 4.3.2 *Assume that $(\beta_i, c)_{i=1,2}$ fulfil the conditions (4.1.6), (4.1.7), (4.1.8), (4.1.9), (4.1.10), (4.1.11), (4.1.12) and (4.1.13). Then, there exists a constant $K > 0$ independent of n such that*

$$\|u_n\|^2_{H^{1,2}(\Omega_n)} \leq K \|f_n\|^2_{L^2(\Omega_n)} \leq K \|f\|^2_{L^2(\Omega)}.$$

In order to prove Theorem 4.3.2, we need some preliminary results.

Lemma 4.3.4 *For every $\epsilon > 0$ satisfying $\varphi(t) \leq \epsilon$, there exists a constant $C > 0$ independent of n, such that*

$$\left\|\partial_x^j u_n\right\|_{L^2(\Omega_n)}^2 \leq C\epsilon^{2(2-j)} \left\|\partial_x^2 u_n\right\|_{L^2(\Omega_n)}^2, \; j = 0,\, 1.$$

Proof. Replacing in Lemma 4.2.2 v by u_n and $]a, b[$ by $]\varphi_1(t), \varphi_2(t)[$, for a fixed t, we obtain

$$\begin{aligned} \int_{\varphi_1(t)}^{\varphi_2(t)} \left(\partial_x^j u_n\right)^2 dx &\leq C\varphi(t)^{2(2-j)} \int_{\varphi_1(t)}^{\varphi_2(t)} \left(\partial_x^2 u_n\right)^2 dx \\ &\leq C\epsilon^{2(2-j)} \int_{\varphi_1(t)}^{\varphi_2(t)} \left(\partial_x^2 u_n\right)^2 dx \end{aligned}$$

where C is the constant of Lemma 4.2.2. Integrating with respect to t, we obtain the desired estimates. ■

Proposition 4.3.1 *There exists a constant $C > 0$ independent of n such that*

$$\|\partial_t u_n\|_{L^2(\Omega_n)}^2 + \left\|\partial_x^2 u_n\right\|_{L^2(\Omega_n)}^2 \leq C \|f\|_{L^2(\Omega)}^2.$$

Then, Theorem 4.3.2 is a direct consequence of Lemma 4.3.4 and Proposition 4.3.1, since ϵ is independent of n.

Proof. In order to prove Proposition 4.3.1, we develop the inner product in $L^2(\Omega_n)$

$$\begin{aligned} \|f_n\|_{L^2(\Omega_n)}^2 &= \langle \partial_t u_n - c^2\partial_x^2 u_n, \partial_t u_n - c^2\partial_x^2 u_n \rangle \\ &= \|\partial_t u_n\|_{L^2(\Omega_n)}^2 + \|c^2.\partial_x^2 u_n\|_{L^2(\Omega_n)}^2 - 2\langle \partial_t u_n, c^2\partial_x^2 u_n\rangle. \end{aligned}$$

Calculating the last term of the previous relation, we obtain

$$\begin{aligned} \langle \partial_t u_n, c^2\partial_x^2 u_n\rangle &= \int_{\Omega_n} \partial_t u_n . c^2\partial_x^2 u_n \; dt\; dx \\ &= -\int_{\Omega_n} c^2\partial_x\partial_t u_n.\partial_x u_n \; dt\; dx + \int_{\partial\Omega_n} c^2\partial_t u_n.\partial_x u_n \nu_x \; d\sigma. \end{aligned}$$

So,

$$\begin{aligned}
-2\left\langle \partial_t u_n, c^2\partial_x^2 u_n\right\rangle &= \int_{\Omega_n} c^2\partial_t\left(\partial_x u_n\right)^2 dt\ dx - 2\int_{\partial\Omega_n} c^2\partial_t u_n.\partial_x u_n \nu_x\ d\sigma \\
&= -\int_{\Omega_n} 2cc'\left(\partial_x u_n\right)^2 dt\ dx + \int_{\partial\Omega_n} c^2\left(\partial_x u_n\right)^2 \nu_t\ d\sigma \\
&\quad -2\int_{\partial\Omega_n} c^2\partial_t u_n.\partial_x u_n \nu_x\ d\sigma \\
&= \int_{\partial\Omega_n} c^2\left[\left(\partial_x u_n\right)^2\nu_t - 2\partial_t u_n.\partial_x u_n\nu_x\right] d\sigma \\
&\quad -\int_{\Omega_n} 2cc'\left(\partial_x u_n\right)^2 dt\ dx
\end{aligned}$$

with ν_t, ν_x are the components of the outward normal vector at the boundary of Ω_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of Ω_n where $t = a_n$, we have $u_n = 0$ and consequently $\partial_x u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary of Ω_n where $t = T$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$A = \int_{\varphi_1(T)}^{\varphi_2(T)} c^2\left(\partial_x u_n\right)^2 dx$$

is nonnegative. On the parts of the boundary where $x = \varphi_i(t)$, $i = 1, 2$, we have $\nu_x = \dfrac{(-1)^i}{\sqrt{1+(\varphi_i')^2(t)}}$, $\nu_t = \dfrac{(-1)^{i+1}\varphi_i'(t)}{\sqrt{1+(\varphi_i')^2(t)}}$ and

$$\partial_x u_n\left(t, \varphi_i(t)\right) + \beta_i(t)\, u_n\left(t, \varphi_i(t)\right) = 0, i = 1, 2.$$

Consequently the corresponding integral is

$$\begin{aligned}
&\int_{a_n}^{T} c^2\varphi_1'(t)\left[\partial_x u_n\left(t, \varphi_1(t)\right)\right]^2 dt - 2\int_{a_n}^{T}\left(\beta_1 c^2\right)(t)\,\partial_t u_n\left(t, \varphi_1(t)\right).u_n\left(t, \varphi_1(t)\right) dt \\
&-\int_{a_n}^{T} c^2\varphi_2'(t)\left[\partial_x u_n\left(t, \varphi_2(t)\right)\right]^2 dt + 2\int_{a_n}^{T}\left(\beta_2 c^2\right)(t)\,\partial_t u_n\left(t, \varphi_2(t)\right).u_n\left(t, \varphi_2(t)\right) dt.
\end{aligned}$$

By setting

$$\begin{aligned}
I_{n,k} &= (-1)^{k+1}\int_{a_n}^{T} c^2\varphi_k'(t)\left[\partial_x u_n\left(t, \varphi_k(t)\right)\right]^2 dt,\ \ k = 1, 2, \\
J_{n,k} &= (-1)^k\, 2\int_{a_n}^{T}\left(\beta_k c^2\right)(t)\,\partial_t u_n\left(t, \varphi_k(t)\right).u_n\left(t, \varphi_k(t)\right) dt,\ \ k = 1, 2,
\end{aligned}$$

we have

$$-2\left\langle \partial_t u_n, c^2\partial_x^2 u_n\right\rangle \geq -\left|I_{n,1}\right| - \left|I_{n,2}\right| - \left|J_{n,1}\right| - \left|J_{n,2}\right| - \int_{\Omega_n} 2cc'\left(\partial_x u_n\right)^2 dtdx. \qquad (4.3.4)$$

1) Estimation of $I_{n,k}, k = 1, 2$

Lemma 4.3.5 *There exists a constant $K > 0$ independent of n such that*

$$|I_{n,k}| \leq K\epsilon \left\|\partial_x^2 u_n\right\|^2_{L^2(\Omega_n)}, \quad k = 1, 2.$$

Proof. We convert the boundary integral $I_{n,1}$ into a surface integral by setting

$$\begin{aligned}
\left[\partial_x u_n(t, \varphi_1(t))\right]^2 &= -\frac{\varphi_2(t) - x}{\varphi_2(t) - \varphi_1(t)} \left[\partial_x u_n(t, x)\right]^2 \Big|_{x=\varphi_1(t)}^{x=\varphi_2(t)} \\
&= -\int_{\varphi_1(t)}^{\varphi_2(t)} \frac{\partial}{\partial x} \left\{ \frac{\varphi_2(t) - x}{\varphi(t)} \left[\partial_x u_n(t, x)\right]^2 \right\} dx \\
&= -2 \int_{\varphi_1(t)}^{\varphi_2(t)} \frac{\varphi_2(t) - x}{\varphi(t)} \partial_x u_n(t, x)\, \partial_x^2 u_n(t, x)\, dx \\
&\quad + \int_{\varphi_1(t)}^{\varphi_2(t)} \frac{1}{\varphi(t)} \left[\partial_x u_n(t, x)\right]^2 dx.
\end{aligned}$$

Then, we have

$$\begin{aligned}
I_{n,1} &= \int_{a_n}^{T} c^2(t)\, \varphi_1'(t) \left[\partial_x u_n(t, \varphi_1(t))\right]^2 \, dt \\
&= \int_{\Omega_n} c^2(t) \frac{\varphi_1'(t)}{\varphi(t)} (\partial_x u_n)^2 \, dt \, dx \\
&\quad -2 \int_{\Omega_n} c^2(t) \frac{\varphi_2(t) - x}{\varphi(t)} \varphi_1'(t) (\partial_x u_n)(\partial_x^2 u_n) \, dt \, dx.
\end{aligned}$$

Thanks to Lemma 4.3.4, we can write

$$\int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_x u_n(t, x)\right]^2 dx \leq C\varphi(t)^2 \int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_x^2 u_n(t, x)\right]^2 dx.$$

Therefore

$$\int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_x u_n(t, x)\right]^2 \frac{|\varphi_1'|}{\varphi} dx \leq C |\varphi_1'| \varphi \int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_x^2 u_n(t, x)\right]^2 dx,$$

consequently

$$\begin{aligned}
|I_{n,1}| &\leq C \int_{\Omega_n} c^2(t) |\varphi_1'| \varphi (\partial_x^2 u_n)^2 \, dt \, dx \\
&\quad + 2 \int_{\Omega_n} c^2(t) |\varphi_1'| |\partial_x u_n| |\partial_x^2 u_n| \, dt dx,
\end{aligned}$$

since $\left|\frac{\varphi_2(t) - x}{\varphi(t)}\right| \leq 1$. So, for all $\epsilon > 0$, we have

$$\begin{aligned}
|I_{n,1}| &\leq C \int_{\Omega_n} c^2(t) |\varphi_1'| \varphi (\partial_x^2 u_n)^2 \, dt \, dx \\
&\quad + \int_{\Omega_n} \epsilon\, c^2(t) (\partial_x^2 u_n)^2 \, dt \, dx + \frac{1}{\epsilon} \int_{\Omega_n} c^2(t) (\varphi_1')^2 (\partial_x u_n)^2 \, dt \, dx.
\end{aligned}$$

Lemma 4.3.4 yields

$$\frac{1}{\epsilon}\int_{\Omega_n} c^2(t)(\varphi_1')^2(\partial_x u_n)^2\,dtdx \;\leq\; C\frac{1}{\epsilon}\int_{\Omega_n} c^2(t)(\varphi_1')^2\varphi^2(\partial_x^2 u_n)^2\,dt\;dx.$$

Thus, there exists a constant $M>0$ independent of n such that

$$\begin{aligned}|I_{n,1}| &\leq C\int_{\Omega_n} c^2(t)\left[|\varphi_1'|\,|\varphi|+\frac{1}{\epsilon}(\varphi_1')^2|\varphi|^2\right](\partial_x^2 u_n)^2\,dt\;dx\\ &\quad+\int_{\Omega_n} c^2(t)\;\epsilon\,(\partial_x^2 u_n)^2\,dt\;dx\\ &\leq M\,\epsilon\int_{\Omega_n}(\partial_x^2 u_n)^2\,dt\;dx,\end{aligned}$$

since $|\varphi_1'\varphi|\leq\epsilon$ and $c^2(t)$ is bounded. The inequality

$$|I_{n,2}| \;\leq\; K\epsilon\,\|\partial_x^2 u_n\|^2_{L^2(\Omega_n)}$$

can be proved by a similar argument. ■

2) Estimation of $J_{n,k}, k=1,2$. We have

$$\begin{aligned}J_{n,1} &= -2\int_{a_n}^{T}(\beta_1 c^2)(t)\,\partial_t u_n(t,\varphi_1(t))\,.u_n(t,\varphi_1(t))\,dt\\ &= -\int_{a_n}^{T}(\beta_1 c^2)(t)\left[\partial_t u_n^2(t,\varphi_1(t))\right]dt.\end{aligned}$$

By setting

$$h(t)=u_n^2(t,\varphi_1(t)),$$

we obtain

$$\begin{aligned}J_{n,1} &= -\int_{a_n}^{T}\beta_1 c^2.\left[h'(t)-\varphi_1'(t)\,\partial_x u_n^2(t,\varphi_1(t))\right]dt\\ &= -\beta_1 c^2.h(t)]_{a_n}^{T}+\int_{a_n}^{T}(\beta_1 c^2)'.h(t)\,dt+\int_{a_n}^{T}\beta_1 c^2.\varphi_1'(t)\,\partial_x u_n^2(t,\varphi_1(t))\,dt.\end{aligned}$$

Thanks to (4.1.8), (4.1.9) and the fact that $u_n^2(a_n,\varphi_1(a_n))=0$, we have

$$-\beta_1 c^2.h(t)\big]_{a_n}^{T}+\int_{a_n}^{T}\left(\beta_1 c^2\right)'.h(t)\,dt\geq 0.$$

The last boundary integral in the expression of $J_{n,1}$ can be treated by a similar argument used in Lemma 4.3.5. So, we obtain the existence of a positive constant K independent of n, such that

$$\left|\int_{a_n}^{T}\beta_1 c^2.\varphi_1'(t)\,\partial_x u_n^2(t,\varphi_1(t))\,dt\right|\leq K\epsilon\left\|\partial_x^2 u_n\right\|^2_{L^2(\Omega_n)}. \tag{4.3.5}$$

By a similar method, we obtain

$$\begin{aligned} J_{n,2} &= \beta_2(t)\,c^2(t)\,u_n^2(t,\varphi_2(t))\big]_{a_n}^T - \int_{a_n}^T (\beta_2 c^2)'.u_n^2(t,\varphi_2(t))\,dt \\ &\quad -\int_{a_n}^T \beta_2 c^2.\varphi_2'(t)\,\partial_x u_n^2(t,\varphi_2(t))\,dt. \end{aligned}$$

Thanks to (4.1.8), (4.1.10) and the fact that $u_n^2(a_n,\varphi_2(a_n))=0$, we have

$$\beta_2(t)\,c^2(t)\,u_n^2(t,\varphi_2(t))\big]_{a_n}^T - \int_{a_n}^T \left(\beta_2 c^2\right)'.u_n^2(t,\varphi_2(t))\,dt \geq 0.$$

Then

$$\left|-\int_{a_n}^T \beta_2 c^2.\varphi_2'(t)\,\partial_x u_n^2(t,\varphi_2(t))\,dt\right| \leq K\epsilon\left\|\partial_x^2 u_n\right\|_{L^2(\Omega_n)}^2 \tag{4.3.6}$$

where K is a positive constant independent of n.

Now, we can complete the proof of Proposition 4.3.1. Summing up the estimates (4.1.7), (4.3.4), (4.3.5) and (4.3.6), and making use of Lemma 4.3.4, we then obtain

$$\begin{aligned} \|f_n\|_{L^2(\Omega_n)}^2 &\geq \|\partial_t u_n\|_{L^2(\Omega_n)}^2 + \|c^2\partial_x^2 u_n\|_{L^2(\Omega_n)}^2 - K_2\|\partial_x^2 u_n\|_{L^2(\Omega_n)}^2 \\ &\quad -K\epsilon\|\partial_x^2 u_n\|_{L^2(\Omega_n)}^2 \\ &\geq \|\partial_t u_n\|_{L^2(\Omega_n)}^2 + (d_1^2 - K\epsilon - K_2)\|\partial_x^2 u_n\|_{L^2(\Omega_n)}^2 \end{aligned}$$

where K_2 is a positive number such that $2|cc'| \leq K_2$. Then, it is sufficient to choose ϵ such that $(d_1^2 - K\epsilon - K_2) > 0$, to get a constant $K_0 > 0$ independent of n such that

$$\|f_n\|_{L^2(\Omega_n)}^2 \geq K_0\left(\|\partial_t u_n\|_{L^2(\Omega_n)}^2 + \left\|\partial_x^2 u_n\right\|_{L^2(\Omega_n)}^2\right).$$

But

$$\|f_n\|_{L^2(\Omega_n)} \leq \|f\|_{L^2(\Omega)},$$

then, there exists a constant $C > 0$, independent of n satisfying

$$\|\partial_t u_n\|_{L^2(\Omega_n)}^2 + \left\|\partial_x^2 u_n\right\|_{L^2(\Omega_n)}^2 \leq C\|f_n\|_{L^2(\Omega_n)}^2 \leq C\|f\|_{L^2(\Omega)}^2.$$

This ends the proof of Proposition 4.3.1.

Passage to the limit We are now in position to prove the main result of this work.

Theorem 4.3.3 *Assume that the following conditions are satisfied*

(1) $(\varphi_i)_{i=1,2}$ *fulfil the assumptions (4.1.1) and (4.1.2).*

(2) the coefficient c *verifies the conditions (4.1.6) and (4.1.7).*

(3) $(\beta_i)_{i=1,2}$ *fulfil the conditions (4.1.8), (4.1.11) and (4.1.12).*

(4) $(\varphi_i, \beta_i, c)_{i=1,2}$ *fulfil the conditions (4.1.13), (4.1.9) and (4.1.10).*

Then, for T *small enough, Problem (4.1.5) admits a (unique) solution* u *belonging to*

$$H_\gamma^{1,2}(\Omega) = \left\{ u \in H^{1,2}(\Omega) ; (\partial_x u + \beta_i(t) u)_{/\Gamma_i} = 0, \ i = 1, 2 \right\},$$

where Γ_i, $i = 1, 2$ *are the parts of the boundary of* Ω *where* $x = \varphi_i(t)$*.*

Proof. Choose a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of the domains defined above, such that $\Omega_n \subseteq \Omega$ with (a_n) a decreasing sequence to 0, as $n \to \infty$. Then, we have $\Omega_n \to \Omega$, as $n \to \infty$.

Consider the solution $u_n \in H^{1,2}(\Omega_n)$ of the Robin boundary value problem

$$\begin{cases} \partial_t u_n - c^2(t)\,\partial_x^2 u_n = f \quad \text{in } \Omega_n \\ u_{n/t=a_n} = 0 \\ \partial_x u_n + \beta_i(t)\, u_{n/\Gamma_{n,i}} = 0, \, i = 1, \, 2, \end{cases}$$

where $\Gamma_{n,i}$ are the parts of the boundary of Ω_n where $x = \varphi_i(t)$, $i = 1, 2$. Such a solution u_n exists by Theorem 4.3.1. Let $\widetilde{u_n}$ the $0-$extension of u_n to Ω. In virtue of Theorem 4.3.2, we know that there exists a constant $K > 0$ such that

$$\|\widetilde{u_n}\|^2_{L^2(\Omega)} + \left\|\widetilde{\partial_t u_n}\right\|^2_{L^2(\Omega)} + \left\|\widetilde{\partial_x u_n}\right\|^2_{L^2(\Omega)} + \left\|\widetilde{\partial_x^2 u_n}\right\|^2_{L^2(\Omega)} \leq K \|f\|^2_{L^2(\Omega)}.$$

This means that $\widetilde{u_n}$, $\widetilde{\partial_t u_n}$, $\widetilde{\partial_x^j u_n}$, for $j = 1, 2$ are bounded functions in $L^2(\Omega)$. So, for a suitable increasing sequence of integers n_k, $k = 1, 2, \ldots$, there exist functions

$$u, \, v \text{ and } v_j, \, j = 1, \, 2$$

in $L^2(\Omega)$ such that

$$\begin{array}{lcll} \widetilde{u_{n_k}} & \rightharpoonup & u & \text{weakly in } L^2(\Omega), \quad k \to \infty \\ \widetilde{\partial_t u_{n_k}} & \rightharpoonup & v & \text{weakly in } L^2(\Omega), \quad k \to \infty \\ \widetilde{\partial_x^j u_{n_k}} & \rightharpoonup & v_j & \text{weakly in } L^2(\Omega), \quad k \to \infty, \, j = 1, \, 2. \end{array}$$

Let then $\theta \in D(\Omega)$. For n_k large enough we have supp $\theta \subset \Omega_{n_k}$. Thus

$$\begin{aligned}
\langle v, \theta \rangle_{D'(\Omega)\times D(\Omega)} &= \lim_{n_k \longrightarrow \infty} \int_\Omega \widetilde{\partial_t u_{n_k}}.\theta \; dt \; dx \\
&= \lim_{n_k \longrightarrow \infty} \int_{\Omega_{n_k}} \partial_t u_{n_k}.\theta \; dt \; dx \\
&= \lim_{n_k \longrightarrow \infty} \langle \partial_t u_{n_k}, \theta \rangle_{D'\left(\Omega_{n_k}\right)\times D\left(\Omega_{n_k}\right)} \\
&= -\lim_{n_k \longrightarrow \infty} \langle u_{n_k}, \partial_t \theta \rangle_{D'\left(\Omega_{n_k}\right)\times D\left(\Omega_{n_k}\right)} \\
&= -\lim_{n_k \longrightarrow \infty} \int_\Omega \widetilde{u_{n_k}}.\partial_t \theta \; dt \; dx \\
&= -\lim_{n_k \longrightarrow \infty} \langle \widetilde{u_{n_k}}, \partial_t \theta \rangle_{D'(\Omega)\times D(\Omega)} \\
&= -\langle u, \partial_t \theta \rangle_{D'(\Omega)\times D(\Omega)} \\
&= \langle \partial_t u, \theta \rangle_{D'(\Omega)\times D(\Omega)}.
\end{aligned}$$

Then, $v = \partial_t u$ in $D'(\Omega)$ and so in $L^2(\Omega)$. By a similar manner, we prove that

$$v_1 = \partial_x u \text{ and } v_2 = \partial_x^2 u$$

in the sense of distributions in Ω and so in $L^2(\Omega)$. Finally, $u \in H^{1,2}(\Omega)$. On the other hand,

$$\partial_t u_{n_k} - c^2(t)\, \partial_x^2 u_{n_k} = f_{n_k} = f_{/\Omega_{n_k}}$$

and

$$\widetilde{\partial_t u_{n_k}} - c^2\, \widetilde{(t)\, \partial_x^2 u_{n_k}} = \widetilde{f_{n_k}}.$$

But

$$\widetilde{f_{n_k}} \longrightarrow f \text{ in } L^2(\Omega)$$

and

$$\widetilde{\partial_t u_{n_k}} - c^2\, \widetilde{(t)\, \partial_x^2 u_{n_k}} \rightharpoonup \partial_t u - c^2(t)\, \partial_x^2 u.$$

So, we have

$$\partial_t u - c^2(t)\, \partial_x^2 u = f \text{ in } \Omega.$$

On the other hand, the solution u satisfies the boundary conditions

$$\begin{cases} u = 0 \text{ at } t = 0 \\ \partial_x u + \beta_i(t)\, u_{/\Gamma_i} = 0, \; i = 1, 2, \end{cases}$$

since

$$\forall n \in \mathbb{N}, \qquad u_{/\Omega_n} = u_n.$$

This proves the existence of solution to Problem (4.1.5).

The uniqueness of the solution is easy to check, thanks to the hypothesis (4.1.13). ■

4.4 Global in time result

Assume that Ω satisfies (4.1.1). In the case where T is not in the neighborhood of zero, we set $\Omega = D_1 \cup D_2 \cup \Gamma_{T_1}$ where

$$D_1 = \left\{(t,x) \in \mathbb{R}^2 : 0 < t < T_1,\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

$$D_2 = \left\{(t,x) \in \mathbb{R}^2 : T_1 < t < T,\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

$$\Gamma_{T_1} = \left\{(T_1,x) \in \mathbb{R}^2 :\ \varphi_1(T_1) < x < \varphi_2(T_1)\right\}$$

with T_1 small enough.

In the sequel, f stands for an arbitrary fixed element of $L^2(\Omega)$ and $f_i = f_{/D_i}$, $i = 1, 2$.

Theorem 4.3.3 applied to the triangular domain D_1, shows that there exists a unique solution $u_1 \in H^{1,2}(D_1)$ of the problem

$$\begin{cases} \partial_t u_1 - c^2(t)\,\partial_x^2 u_1 = f_1 \in L^2(D_1) \\ \\ \partial_x u_1 + \beta_i(t)\, u_{1/\Gamma_{i,1}} = 0,\ i = 1,\ 2, \end{cases} \tag{4.4.1}$$

with $\Gamma_{i,1}$ are the parts of the boundary of D_1 where $x = \varphi_i(t)$, $i = 1, 2$.

Lemma 4.4.1 *If $u \in H^{1,2}(D_2)$, then $u_{/\Gamma_{T_1}} \in H^1(\Gamma_{T_1})$, $u_{/x=\varphi_1(t)} \in H^{\frac{3}{4}}(\Gamma_{1,2})$ and $u_{/x=\varphi_2(t)} \in H^{\frac{3}{4}}(\Gamma_{2,2})$, where $\Gamma_{i,2}$ are the parts of the boundary of D_2 where $x = \varphi_i(t)$, $i = 1, 2$.*

The above lemma follows from Lemma 2.1.1 of Chapter 2 by using the transformation

$$(t,x) \longmapsto (t',x') = (t, \varphi(t)\,x + \varphi_1(t)).$$

Hereafter, we denote the trace $u_{1/\Gamma_{T_1}}$ by ψ which is in the Sobolev space $H^1(\Gamma_{T_1})$ because $u_1 \in H^{1,2}(D_1)$ (see Lemma 4.4.1).

Now, consider the following problem in D_2

$$\begin{cases} \partial_t u_2 - c^2(t)\,\partial_x^2 u_2 = f_2 \ \in L^2(D_2) \\ u_{2/\Gamma_{T_1}} = \psi \\ \partial_x u_2 + \beta_i(t)\, u_{2/\Gamma_{i,2}} = 0,\ i = 1,\ 2, \end{cases} \tag{4.4.2}$$

with $\Gamma_{i,2}$ are the parts of the boundary of D_2 where $x = \varphi_i(t)$, $i = 1, 2$.

We use the following result, which is a consequence of [43, Theorem 4.3, Vol. 2], to solve Problem (4.4.2).

Proposition 4.4.1 *Let Q be the rectangle $]0,T[\times]0,1[$, $f \in L^2(Q)$ and $\psi \in H^1(\gamma_0)$. Then, the problem*

$$\begin{cases} \partial_t u - c^2(t)\,\partial_x^2 u = f \ \text{in } Q \\ u_{/\gamma_0} = \psi \\ \partial_x u + \beta_i(t)\, u_{/\gamma_i} = 0,\ i = 1,\ 2, \end{cases}$$

where $\gamma_0 = \{0\}\times]0,1[$, $\gamma_1 =]0,T[\times\{0\}$ and $\gamma_2 =]0,T[\times\{1\}$, admits a (unique) solution $u \in H^{1,2}(Q)$.

Remark 4.4.1 *In the application of [43, Theorem 4.3, Vol.2], we can observe that there are no compatibility conditions to satisfy because $\partial_x\psi$ is only in $L^2(\gamma_0)$.*

Thanks to the transformation

$$(t,x) \longmapsto (t,y) = (t, \varphi(t)\,x + \varphi_1(t)),$$

we deduce the following result.

Proposition 4.4.2 *Problem (4.4.2) admits a (unique) solution $u_2 \in H^{1,2}(D_2)$.*

So, the function u defined by

$$u = \begin{cases} u_1 \text{ in } D_1 \\ u_2 \text{ in } D_2 \end{cases}$$

is the (unique) solution of Problem (4.1.5) for an arbitrary T. Our second main result is as follows.

Theorem 4.4.1 *Assume that the following conditions are satisfied*

(1) $(\varphi_i)_{i=1,2}$ *fulfil the assumptions (4.1.1) and (4.1.2).*

(2) the coefficient c *verifies the conditions (4.1.6) and (4.1.7).*

(3) $(\beta_i)_{i=1,2}$ *fulfil the conditions (4.1.8), (4.1.11) and (4.1.12).*

(4) $(\varphi_i, \beta_i, c)_{i=1,2}$ *fulfil the conditions (4.1.13), (4.1.9) and (4.1.10).*

Then, Problem (4.1.5) admits a (unique) solution u *belonging to*

$$H_{\gamma}^{1,2}(\Omega) = \left\{ u \in H^{1,2}(\Omega) ; (\partial_x u + \beta_i(t) u)_{/\Gamma_i} = 0,\ i = 1, 2 \right\},$$

where Γ_i, $i = 1,\ 2$ *are the parts of the boundary of* Ω *where* $x = \varphi_i(t)$.

Remark 4.4.2 *Using the same method in the case where* $\varphi(T) = 0$ *we can obtain a result similar to Theorem 4.4.1.*