CHAPTER 3

Parabolic equation with Cauchy-Dirichlet boundary conditions in a non-regular domain of $\mathbb{R}^3$

Abstract. In this work we give new results of existence, uniqueness and maximal regularity of a solution to a parabolic equation set in a non-regular domain

$$Q = \left\{(t, x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\right\} \times]0, b[$$

of $\mathbb{R}^3$, with Cauchy-Dirichlet boundary conditions, under some assumptions on the functions $(\varphi_i)_{i=1,2}$. The right hand side term of the equation is taken in $L^2(Q)$. The method used is based on the approximation of the domain Q by a sequence of sub-domains $(Q_n)_n$ which can be transformed into regular domains. This work is an extension of the one space variable case studied in [58].

Key words. Parabolic equations, non-regular domains, anisotropic Sobolev spaces.

3.1 Introduction

Let Ω be an open set of $\mathbb{R}^2$ defined by

$$\Omega = \left\{(t, x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\right\}$$

where T is a finite positive number, while φ_1 and φ_2 are continuous real-valued functions defined on $[0, T]$, Lipschitz continuous on $]0, T[$, and such that

$$\varphi_1(t) < \varphi_2(t)$$

for $t \in]0, T[$. φ_1 is allowed to coincide with φ_2 for $t = 0$ and for $t = T$. For a fixed positive number b, let Q be the three-dimensional domain defined by

$$Q = \left\{(t, x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\right\} \times]0, b[,$$

with boundary $\partial Q = (\Gamma \times]0, b[) \cup (\Omega \times \{0\}) \cup (\Omega \times \{b\})$, Γ is the boundary of Ω (see Fig. 6).

In this work, we study the existence and the regularity of the solution of the parabolic equation with Cauchy-Dirichlet boundary conditions

$$\begin{cases} \partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u = f \text{ in } Q \\ \\ u = 0 \text{ on } \partial Q \setminus \Gamma_T, \end{cases} \tag{3.1.1}$$

where Γ_T is the part of the boundary of Q where $t = T$. The right-hand side term f of the equation lies in $L^2(Q)$.

In Baderko [8] we can find domains of the same kind but which can not include our domain. In Sadallah [58] the same problem has been studied for a 2m-parabolic operator in the case of one space variable. Further references on the analysis of parabolic problems in non-cylindrical domains are: Savaré [64], Aref'ev and Bagirov [5], Hoffmann and Lewis [22], Labbas, Medeghri and Sadallah [32], [33], and Alkhutov [3]. There are many other works concerning boundary-value problems in non-smooth domains (see, for example, Grisvard [20] and the references therein).

We are especially interested in the question of what conditions the functions $(\varphi_i)_{i=1,2}$ must verify in order that Problem (3.1.1) has a solution with optimal regularity, that is a solution u belonging to the anisotropic Sobolev space

$$H_0^{1,2}(Q) := \left\{u \in H^{1,2}(Q) : u_{/\partial Q \smallsetminus \Gamma_T} = 0\right\}$$

with

$$H^{1,2}(Q) = \left\{u \in L^2(Q) : \partial_t u, \partial_{x_1}^j u, \partial_{x_2}^j u, \partial_{x_1}\partial_{x_2} u \in L^2(Q), j = 1, 2\right\} ?$$

An idea to solve Problem (3.1.1) consists in transforming the parabolic equation in the non-regular domain Q into a variable-coefficient equation in a regular domain. However, in order to perform this, one must assume that $\varphi_1(0) < \varphi_2(0)$ and $\varphi_1(T) < \varphi_2(T)$. So, in Section 3.2, we prove that Problem (3.1.1) admits a (unique) solution when Q could be transformed into a regular domain by means of a regular change of variable, i.e., we suppose that $\varphi_1(0) < \varphi_2(0)$ and $\varphi_1(T) < \varphi_2(T)$. In Section 3.3 we approximate Q by a sequence (Q_{α_n}) of such domains and we establish an uniform estimate of the type

$$\|u_n\|_{H^{1,2}(Q_{\alpha_n})} \leq K \|f\|_{L^2(Q_{\alpha_n})},$$

where u_n is the solution of Problem (3.1.1) in Q_{α_n} and K is a constant independent of n. Finally, in Section 3.4 we take limits in (Q_{α_n}) in order to reach the domain Q.

The main assumptions on the functions $(\varphi_i)_{i=1,2}$ are

$$\varphi_i'(t)(\varphi_2(t) - \varphi_1(t)) \longrightarrow 0 \quad \text{as } t \longrightarrow 0, \quad i = 1, 2, \tag{3.1.2}$$

and

$$\varphi_i'(t)(\varphi_2(t) - \varphi_1(t)) \longrightarrow 0 \quad \text{as } t \longrightarrow T, \quad i = 1, 2. \tag{3.1.3}$$

3.2 Resolution of the problem in a reference domain

In this section, we replace Q by

$$Q_\alpha = \left\{(t, x_1) \in \mathbb{R}^2 : \alpha < t < T - \alpha; \varphi_1(t) < x_1 < \varphi_2(t)\right\} \times]0, b[,$$

with $\alpha > 0$. Thus, we have

$$\begin{cases} \varphi_1(\alpha) < \varphi_2(\alpha) \\ \varphi_1(T-\alpha) < \varphi_2(T-\alpha), \end{cases}$$

(see Fig. 6).

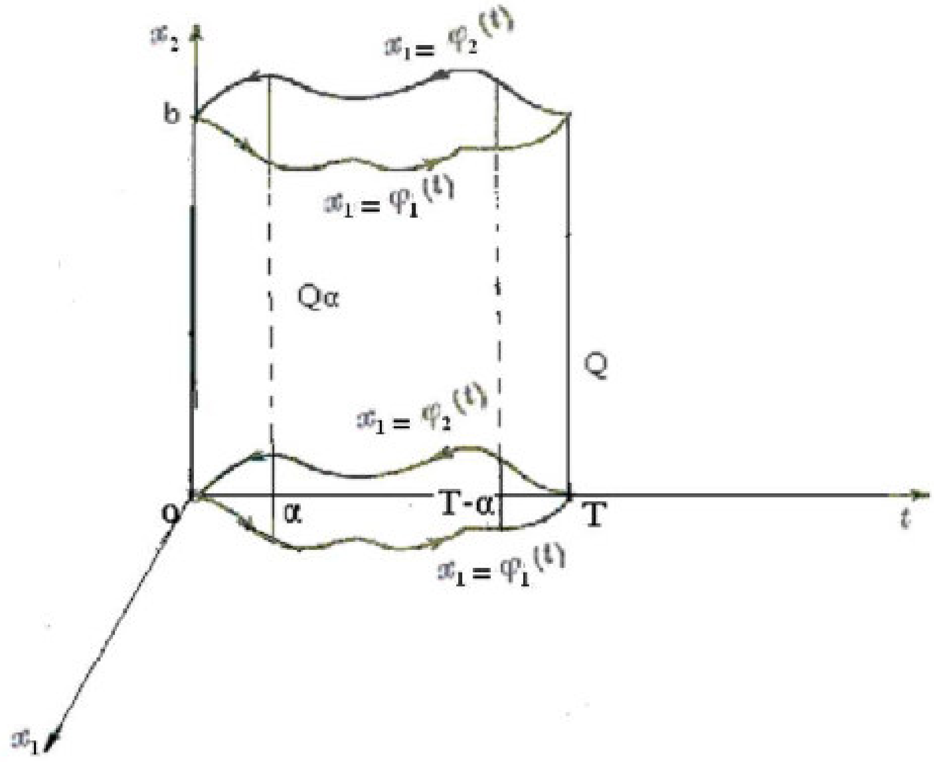

Fig. 6 : The non-regular domains Q and Q_α.

We can find a change of variable ψ mapping Q_α into the parallelepiped

$$P_\alpha =]\alpha, T-\alpha[\times]0,1[\times]0,b[,$$

which leaves the variable t unchanged. ψ is defined as follows:

$$\begin{array}{llllll} \psi: & Q_\alpha & \longrightarrow & P_\alpha & & \\ & (t,x_1,x_2) & \longmapsto & \psi(t,x_1,x_2) = (\tau,y_1,y_2) & = & \left(t, \dfrac{x_1-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}, x_2\right). \end{array}$$

The mapping ψ transforms the parabolic equation in the domain Q_α into a variable-coefficient parabolic equation in the parallelepiped P_α. Indeed, the equation

$$\partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u = f$$

in Q_α is equivalent to the following

$$\partial_\tau v + a(\tau, y_1)\,\partial_{y_1} v - c(\tau)\,\partial_{y_1}^2 v - \partial_{y_2}^2 v = g$$

in P_α, where a and c are defined by

$$a(\tau, y_1) = \frac{(\varphi_1'(\tau) - \varphi_2'(\tau))\, y_1 - \varphi_1'(\tau)}{\varphi_2(\tau) - \varphi_1(\tau)},$$
$$c(\tau) = \frac{1}{(\varphi_2(\tau) - \varphi_1(\tau))^2}$$

and

$$g(\tau, y_1, y_2) = f(t, x_1, x_2),$$
$$v(\tau, y_1, y_2) = u(t, x_1, x_2).$$

Since the functions a, c and $\varphi_2 - \varphi_1$ are bounded, it is easy to check the following

Lemma 3.2.1 *$u \in H^{1,2}(Q_\alpha)$ if and only if $v \in H^{1,2}(P_\alpha)$.*

Proof. The mapping ψ is tri-Lipschitz and therefore it preserves the Sobolev spaces $H^{1,2}$. ∎

The boundary conditions on v which correspond to the boundary conditions on u are the following

$$v_{/\partial P_\alpha \setminus \Gamma_{T-\alpha}} = 0,$$

where $\Gamma_{T-\alpha}$ is the part of the boundary of P_α where $t = T - \alpha$.

In the sequel, the variables (τ, y_1, y_2) will be denoted again by (t, x_1, x_2).

Theorem 3.2.1 *The operator*

$$L' = \partial_t + a\partial_{x_1} - c\partial_{x_1}^2 - \partial_{x_2}^2$$

is an isomorphism from $H_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$, with

$$H_0^{1,2}(P_\alpha) = \left\{u \in H^{1,2}(P_\alpha) : u_{/\partial P_\alpha \setminus \Gamma_{T-\alpha}} = 0\right\}.$$

Consider the simplified problem

$$\begin{cases} \partial_t v - c(t)\,\partial_{x_1}^2 v - \partial_{x_2}^2 v = g \text{ in } P_\alpha \\ v_{/\partial P_\alpha \smallsetminus \Gamma_{T-\alpha}} = 0. \end{cases} \tag{3.2.1}$$

Note that $g \in L^2(P_\alpha)$ if and only if $f \in L^2(Q_\alpha)$.

Lemma 3.2.2 *For every $g \in L^2(P_\alpha)$, there exists a unique $v \in H_0^{1,2}(P_\alpha)$ solution of Problem (3.2.1).*

Proof. Since the coefficient $c(t)$ is continuous in $\overline{P_\alpha}$, the optimal regularity is given by Ladyzhenskaya-Solonnikov-Ural'tseva [35]. Then in order to apply this classical result, it is important to observe that the operator

$$L_1 = \partial_t - c\partial_{x_1}^2 - \partial_{x_2}^2$$

is uniformly parabolic. Indeed L_1 can be written in the divergential form as

$$L_1 = \partial_t - \sum_{i,j=1}^{2} \partial_{x_i}\left(a_{ij}(t)\,\partial_{x_j}\right)$$

with

$$a_{11}(t) = c(t)\,,\; a_{12}(t) = a_{21}(t) = 0,\; a_{22}(t) = 1.$$

Let $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$, we must prove that there exists

$$\lambda > 0 : \sum_{i,j=1}^{2} a_{ij}(t)\,\zeta_i\zeta_j \geq \lambda\,|\zeta|^2.$$

Set $\lambda = \max\left(\frac{1}{d}, 1\right)$ with d is a strictly positive constant such that

$$(\varphi_2(t) - \varphi_1(t)) \leq d.$$

Then

$$\begin{aligned} \textstyle\sum_{i,j=1}^{2} a_{ij}(t)\,\zeta_i\zeta_j = c\partial_{x_1}^2 - \partial_{x_2}^2 &\geq \frac{1}{d^2}\zeta_1^2 + \zeta_2^2 \\ &\geq \lambda\left[\zeta_1^2 + \zeta_2^2\right] \\ &\geq \lambda\,|\zeta|^2. \end{aligned}$$

■

Lemma 3.2.3 *The operator*

$$\begin{array}{rlcl} B: & H_0^{1,2}(P_\alpha) & \longrightarrow & L^2(P_\alpha) \\ & v & \longmapsto & Bv = a(t,x_1)\,\partial_{x_1} v \end{array}$$

is compact.

Proof. P_α has the "horn property" of Besov [9], so

$$\begin{array}{rlcl} \partial_{x_1}: & H_0^{1,2}(P_\alpha) & \longrightarrow & H^{\frac{1}{2},1}(P_\alpha) \\ & v & \longmapsto & \partial_{x_1} v \end{array}$$

is continuous. Since P_α is bounded, the canonical injection is compact from $H^{\frac{1}{2},1}(P_\alpha)$ into $L^2(P_\alpha)$ (see for instance [9]), where

$$H^{\frac{1}{2},1}(P_\alpha) = L^2\left(\alpha, T-\alpha; H^1(]0,1[\times]0,b[)\right) \cap H^{\frac{1}{2}}\left(\alpha, T-\alpha; L^2(]0,1[\times]0,b[)\right).$$

For the complete definitions of the $H^{r,s}$ Hilbertian Sobolev spaces see for instance [43].

Consider the composition

$$\begin{array}{rlclcl} \partial_{x_1}: & H_0^{1,2}(P_\alpha) & \rightarrow & H^{\frac{1}{2},1}(P_\alpha) & \rightarrow & L^2(P_\alpha) \\ & v & \mapsto & \partial_{x_1} v & \mapsto & \partial_{x_1} v, \end{array}$$

then ∂_{x_1} is a compact operator from $H_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$. Since $a(.,.)$ is a bounded function, the operator $a\partial_{x_1}$ is also compact from $H_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$. ■

Lemma 3.2.2 shows that the operator $L_1 = \partial_t - c\partial_{x_1}^2 - \partial_{x_2}^2$ is an isomorphism from $H_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$, on the other hand the operator $a\partial_{x_1}$ is compact, consequently, $L_1 + a\partial_{x_1}$ is a Fredholm operator from $H_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$. Thus the invertibility of $L_1 + a\partial_{x_1}$ follows from its injectivity.

Let us consider $v \in H_0^{1,2}(P_\alpha)$ a solution of

$$\partial_t v + a\partial_{x_1} v - c\partial_{x_1}^2 v - \partial_{x_2}^2 v = 0 \text{ in } P_\alpha.$$

We perform the inverse change of variable of ψ. Thus we set

$$u = v \circ \psi.$$

It turns out that $u \in H_0^{1,2}(Q_\alpha)$ and

$$\partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u = 0 \text{ in } Q_\alpha.$$

Using Green formula, we have

$$\begin{aligned}\int_{Q_\alpha} \left(\partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u\right) u \; dt \; dx_1 dx_2 &= \int_{\partial Q_\alpha} \left(\tfrac{1}{2}|u|^2 \nu_t - \partial_{x_1} u.u\nu_{x_1} - \partial_{x_2} u.u\nu_{x_2}\right) d\sigma \\ &\quad + \int_{Q_\alpha} \left(|\partial_{x_1} u|^2 + |\partial_{x_2} u|^2\right) dt \; dx_1 dx_2\end{aligned}$$

where ν_t, ν_{x_1}, ν_{x_2} are the components of the unit outward normal vector at ∂Q_α. All the boundary integrals vanish except $\int_{\partial Q_\alpha} |u|^2 \nu_t \; d\sigma$. We have

$$\int_{\partial Q_\alpha} |u|^2 \nu_t d\sigma = \int_{\varphi_1(\alpha)}^{\varphi_2(\alpha)} \int_0^b |u|^2 dx_1 dx_2 + \int_{\varphi_1(T-\alpha)}^{\varphi_2(T-\alpha)} \int_0^b |u|^2 dx_1 dx_2.$$

Then

$$\begin{aligned}\int_{Q_\alpha} \left(\partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u\right) u \; dt \; dx_1 dx_2 &= \tfrac{1}{2}\int_{\varphi_1(\alpha)}^{\varphi_2(\alpha)} \textstyle\int_0^b |u|^2 dx_1 dx_2 + \tfrac{1}{2}\int_{\varphi_1(T-\alpha)}^{\varphi_2(T-\alpha)} \int_0^b |u|^2 dx_1 dx_2 \\ &\quad + \int_{Q_\alpha} \left(|\partial_{x_1} u|^2 + |\partial_{x_2} u|^2\right) dt \; dx_1 dx_2.\end{aligned}$$

Consequently

$$\int_{Q_\alpha} \left(\partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u\right) u \; dt \; dx_1 dx_2 = 0$$

yields the inequality

$$\int_{Q_\alpha} \left(|\partial_{x_1} u|^2 + |\partial_{x_2} u|^2\right) dt \; dx_1 dx_2 \leq 0,$$

because

$$\frac{1}{2}\int_{\varphi_1(\alpha)}^{\varphi_2(\alpha)} \int_0^b |u|^2 dx_1 dx_2 + \frac{1}{2}\int_{\varphi_1(T-\alpha)}^{\varphi_2(T-\alpha)} \int_0^b |u|^2 dx_1 dx_2 \geq 0.$$

This implies that $|\partial_{x_1} u|^2 + |\partial_{x_2} u|^2 = 0$ and consequently $\partial_{x_1}^2 u = \partial_{x_2}^2 u = 0$. Then, the equation of (3.1.1) gives $\partial_t u = 0$. Thus, u is constant. The boundary conditions imply that $u = 0$ in Q_α. This is the desired injectivity.

We shall need the following result in order to justify all the calculus of Section 3.3.

Lemma 3.2.4 *The space*

$$\left\{u \in H^4(P_\alpha);\; u_{/\partial P_\alpha \setminus \Gamma_{T-\alpha}} = 0\right\}$$

is dense in the space

$$\left\{u \in H^{1,2}(P_\alpha);\ u_{/\partial P_\alpha \smallsetminus \Gamma_{T-\alpha}} = 0\right\}.$$

Proof. Let Γ_α the part of the boundary of P_α where $t = \alpha$. Lemma 2.1.2 of Chapter 2 shows that the space

$$\left\{u \in H^{4}(P_\alpha);\ u_{/\partial P_\alpha \smallsetminus \Gamma_{T-\alpha} \smallsetminus \Gamma_\alpha} = 0\right\}$$

is dense in the space

$$\left\{u \in H^{1,2}(P_\alpha);\ u_{/\partial P_\alpha \smallsetminus \Gamma_{T-\alpha} \smallsetminus \Gamma_\alpha} = 0\right\}.$$

So, if

$$u \in \left\{u \in H^{1,2}(P_\alpha);\ u_{/\partial P_\alpha \smallsetminus \Gamma_{T-\alpha} \smallsetminus \Gamma_\alpha} = 0\right\},$$

then there exists a sequence

$$(u_n) \in \left\{u \in H^{4}(P_\alpha);\ u_{/\partial P_\alpha \smallsetminus \Gamma_{T-\alpha} \smallsetminus \Gamma_\alpha} = 0\right\}$$

such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^{1,2}(P_\alpha), \quad n \to \infty.$$

Let (e_n) a sequence of $C^\infty([\alpha, T-\alpha])$ such that

$$e_n(t) = \begin{cases} 1 & \text{if } t \geq \alpha + \dfrac{1}{n}, \\ 0 & \text{if } t \leq \alpha + \dfrac{1}{2n}. \end{cases}$$

The sequence $(e_n u_n)$ belongs to

$$\left\{u \in H^{4}(P_\alpha);\ u_{/\partial P_\alpha \smallsetminus \Gamma_{T-\alpha}} = 0\right\}.$$

In addition

$$e_n u_n \rightharpoonup u \quad \text{weakly in } H^{1,2}(P_\alpha), \quad n \to \infty.$$

■

Remark 3.2.1 *In Lemma 3.2.4, we can replace P_α by Q_α with the help of the change of variable ψ defined above.*

3.3 An uniform estimate

Now we shall prove an uniform estimate which will allow us to take limits in α_n. We denote $u_n \in H^{1,2}(Q_{\alpha_n})$ the solution of Problem (3.1.1) corresponding to a second member $f_n = f_{/Q_{\alpha_n}} \in L^2(Q_{\alpha_n})$ in

$$Q_{\alpha_n} = \Omega_{\alpha_n} \times]0, b[\,,$$

where

$$\Omega_{\alpha_n} = \left\{(t, x_1) \in \mathbb{R}^2 : \alpha_n < t < T - \alpha_n, \varphi_1(t) < x_1 < \varphi_2(t)\right\},$$

with $(\alpha_n)_n$ a sequence decreasing to zero.

Proposition 3.3.1 *There exists a constant K_1 independent of n such that*

$$\|u_n\|_{H^{1,2}(Q_{\alpha_n})} \le K_1 \|f_n\|_{L^2(Q_{\alpha_n})} \le K_1 \|f\|_{L^2(Q)}.$$

In order to prove Proposition 3.3.1, we need some preliminary results.

Lemma 3.3.1 *Let $]\alpha, \beta[\subset \mathbb{R}$. There exists a constant K_2 (independent of α and β) such that*

$$\left\|u^{(j)}\right\|^2_{L^2(]\alpha,\beta[)} \le (\beta - \alpha)^{2(2-j)} K_2 \left\|u^{(2)}\right\|^2_{L^2(]\alpha,\beta[)},\ j = 0, 1,$$

for every $u \in H^2(]\alpha, \beta[) \cap H_0^1(]\alpha, \beta[)$, where $u^{(1)}$ (respectively $u^{(2)}$) is the first (respectively the second) derivative of u on $]\alpha, \beta[$ and $u^{(0)} = u$.

Proof. Consider the particular case where $]\alpha, \beta[=]0, 1[$ and let f an arbitrary fixed element of $L^2(0, 1)$. Then, the solution of the problem

$$\begin{cases} u'' = f \\ u(0) = 0, \\ u(1) = 0, \end{cases}$$

can be written in the form

$$u(y) = \int_0^1 G(x, y) f(y) dy$$

where

$$G(x, y) = \begin{cases} x(y-1) & \text{if } x \le y, \\ y(x-1) & \text{if } y \le x. \end{cases}$$

By using the Cauchy-Schwarz inequality, we obtain the following estimate

$$\|u\|^2_{L^2(]0,1[)} \leq K_2 \|f\|^2_{L^2(]0,1[)}$$

and thus

$$\|u\|^2_{L^2(]0,1[)} \leq K_2 \|u''\|^2_{L^2(]0,1[)}.$$

By a similar argument, we obtain

$$\|u'\|^2_{L^2(]0,1[)} \leq K_2 \|u''\|^2_{L^2(]0,1[)}$$

from the following form of $u'(y)$

$$u'(y) = \int_0^y f(x)\,dx - \int_0^1 \left\{ \int_0^x f(s)\,ds \right\} dx.$$

The general case follows from the previous particular case $]\alpha,\beta[\ = \ ]0,1[$ by an affine change of variable. Indeed, we define the following affine change of variable

$$\begin{array}{lll} [0,1] & \to & [\alpha,\beta] \\ x & \to & (1-x)\alpha + x\beta = y \end{array}$$

and we set

$$u(x) = v(y).$$

Then if $u \in H^2(]0,1[) \cap H^1_0(]0,1[)$, v belongs to $H^2(]\alpha,\beta[) \cap H^1_0(]\alpha,\beta[)$. We have

$$\begin{aligned} \|u'\|^2_{L^2(0,1)} &= \int_0^1 (u')^2(x)\ dx \\ &= \int_\alpha^\beta (v')^2(y)(\beta-\alpha)^2\ \frac{dy}{\beta-\alpha} \\ &= \int_\alpha^\beta (v')^2(y)(\beta-\alpha)\ dy \\ &= (\beta-\alpha)\|v'\|^2_{L^2(]\alpha,\beta[)}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|u''\|^2_{L^2(0,1)} &= \int_0^1 (u'')^2(x)\ dx \\ &= \int_\alpha^\beta (v'')^2(y)(\beta-\alpha)^3\ dy \\ &= (\beta-\alpha)^3\|v''\|^2_{L^2(]\alpha,\beta[)}. \end{aligned}$$

Using the inequality

$$\|u'\|^2_{L^2(0,1)} \leq K_2 \|u''\|^2_{L^2(0,1)}$$

of the previous case, we obtain the desired inequality

$$\|v'\|^2_{L^2(]\alpha,\beta[)} \leq K_2 (\beta-\alpha)^2 \|v''\|^2_{L^2(]\alpha,\beta[)} .$$

The inequality

$$\|v\|^2_{L^2(]\alpha,\beta[)} \leq K_2 (\beta-\alpha)^4 \|v''\|^2_{L^2(]\alpha,\beta[)}$$

can be obtained by a similar method. ■

Lemma 3.3.2 *For every $\epsilon > 0$, chosen such that $(\varphi_2(t) - \varphi_1(t)) \leq \epsilon$, there exists a constant C_1 independent of n such that*

$$\left\|\partial^j_{x_1} u_n\right\|^2_{L^2(Q_{\alpha_n})} \leq C_1 \epsilon^{2(2-j)} \left\|\partial^2_{x_1} u_n\right\|^2_{L^2(Q_{\alpha_n})}, \; j = 0, 1.$$

Proof. Replacing in Lemma 3.3.1 u by u_n and $]\alpha,\beta[$ by $]\varphi_1(t), \varphi_2(t)[$, for a fixed t, we obtain

$$\begin{aligned}\int_{\varphi_1(t)}^{\varphi_2(t)} \left(\partial^j_{x_1} u_n\right)^2 dx_1 &\leq K_2 (\varphi_2(t) - \varphi_1(t))^{2(2-j)} \int_{\varphi_1(t)}^{\varphi_2(t)} \left(\partial^2_{x_1} u_n\right)^2 dx_1 \\ &\leq K_2 \epsilon^{2(2-j)} \int_{\varphi_1(t)}^{\varphi_2(t)} \left(\partial^j_{x_1} u_n\right)^2 dx_1.\end{aligned}$$

Integrating in the previous inequality with respect to t, then with respect to x_2, we get the desired result with $C_1 = K_2$. ■

Proof. of Proposition (3.3.1) Let us denote the inner product in $L^2(Q_{\alpha_n})$ by $\langle .,. \rangle$, then we have

$$\begin{aligned}\|f_n\|^2_{L^2(Q_{\alpha_n})} &= \langle \partial_t u_n - \partial^2_{x_1} u_n - \partial^2_{x_2} u_n, \partial_t u_n - \partial^2_{x_1} u - \partial^2_{x_2} u_n \rangle \\ &= \|\partial_t u_n\|^2_{L^2(Q_{\alpha_n})} + \left\|\partial^2_{x_1} u_n\right\|^2_{L^2(Q_{\alpha_n})} + \left\|\partial^2_{x_2} u_n\right\|^2_{L^2(Q_{\alpha_n})} \\ &\quad -2\langle \partial_t u_n, \partial^2_{x_1} u_n \rangle - 2\langle \partial_t u_n, \partial^2_{x_2} u_n \rangle + 2\langle \partial^2_{x_1} u_n, \partial^2_{x_2} u_n \rangle.\end{aligned}$$

1) Estimation of $-2\langle \partial_t u_n, \partial^2_{x_1} u_n \rangle$ We have

$$\partial_t u_n \partial^2_{x_1} u_n = \partial_{x_1} (\partial_t u_n \partial_{x_1} u_n) - \tfrac{1}{2} \partial_t (\partial_{x_1} u_n)^2 .$$

Then

$$\begin{aligned}-2\langle\partial_t u_n,\partial_{x_1}^2 u_n\rangle &= -2\int_{Q_{\alpha_n}}\partial_t u_n\partial_{x_1}^2 u_n dt\ dx_1dx_2\\ &= -2\int_{Q_{\alpha_n}}\partial_{x_1}\left(\partial_t u_n\partial_{x_1}u_n\right)dt\ dx_1dx_2\\ &\quad+\int_{Q_{\alpha_n}}\partial_t\left(\partial_{x_1}u_n\right)^2 dt\ dx_1dx_2\\ &= \int_{\partial Q_{\alpha_n}}\left[\left(\partial_{x_1}u_n\right)^2\nu_t-2\partial_t u_n\partial_{x_1}u_n\nu_{x_1}\right]d\sigma,\end{aligned}$$

where $\nu_t,\nu_{x_1},\nu_{x_2}$ are the components of the unit outward normal vector at ∂Q_{α_n}. We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Q_{α_n} where $t=\alpha_n$, $x_2=0$ and $x_2=b$ we have $u_n=0$ and consequently $\partial_{x_1}u_n=0$. The correponding boundary integral vanishes. On the part of the boundary where $t=T-\alpha_n$, we have $\nu_{x_1}=0$ and $\nu_t=1$. Accordingly the correponding boundary integral

$$A=\int_0^b\int_{\varphi_1(T-\alpha_n)}^{\varphi_2(T-\alpha_n)}\left(\partial_{x_1}u_n\right)^2dx_1dx_2$$

is nonnegative. On the part of the boundary where $x_1=\varphi_i\left(t\right)$, $i=1,2$, we have $u_n=0$. Differentiating with respect to t we obtain

$$\partial_t u_n=-\varphi_i'\left(t\right)\partial_{x_1}u_n.$$

Consequently, the correponding boundary integral is

$$\begin{aligned}&-\int_0^b\int_{\alpha_n}^{T-\alpha_n}\varphi_1'\left(t\right)\left[\partial_{x_1}u_n\left(t,\varphi_1\left(t\right),x_2\right)\right]^2\ dt\ dx_2\\ &+\int_0^b\int_{\alpha_n}^{T-\alpha_n}\varphi_2'\left(t\right)\left[\partial_{x_1}u_n\left(t,\varphi_2\left(t\right),x_2\right)\right]^2\ dt\ dx_2.\end{aligned}$$

By setting

$$\begin{aligned}I_1 &= -\int_0^b\int_{\alpha_n}^{T-\alpha_n}\varphi_1'\left(t\right)\left[\partial_{x_1}u_n\left(t,\varphi_1\left(t\right),x_2\right)\right]^2\ dt\ dx_2\\ I_2 &= \int_0^b\int_{\alpha_n}^{T-\alpha_n}\varphi_2'\left(t\right)\left[\partial_{x_1}u_n\left(t,\varphi_2\left(t\right),x_2\right)\right]^2\ dt\ dx_2,\end{aligned}$$

we have

$$-2\langle\partial_t u_n,\partial_{x_1}^2 u_n\rangle\geq-\left|I_1\right|-\left|I_2\right|. \tag{3.3.1}$$

■

Lemma 3.3.3 *There exists a constant K_4 independent of n such that*

$$|I_i| \leq K_4\epsilon \left\|\partial_{x_1}^2 u_n\right\|_{L^2(Q_{\alpha_n})}^2, \quad i=1,2.$$

Proof. We convert the boundary integral I_1 into a surface integral by setting

$$\begin{aligned}
\left[\partial_{x_1}u_n\left(t,\varphi_1\left(t\right),x_2\right)\right]^2 &= -\frac{\varphi_2\left(t\right)-x_1}{\varphi_2\left(t\right)-\varphi_1\left(t\right)}\left[\partial_{x_1}u_n\left(t,x_1,x_2\right)\right]^2\Big|_{x_1=\varphi_1(t)}^{x_1=\varphi_2(t)} \\
&= -\int_{\varphi_1(t)}^{\varphi_2(t)}\partial_{x_1}\left\{\frac{\varphi_2\left(t\right)-x_1}{\varphi_2\left(t\right)-\varphi_1\left(t\right)}\left[\partial_{x_1}u_n\right]^2\right\}dx_1 \\
&= -2\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{\varphi_2\left(t\right)-x_1}{\varphi_2\left(t\right)-\varphi_1\left(t\right)}\partial_{x_1}u_n.\partial_{x_1}^2u_n\ dx_1 \\
&\quad +\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{1}{\varphi_2\left(t\right)-\varphi_1\left(t\right)}\left[\partial_{x_1}u_n\right]^2dx_1.
\end{aligned}$$

Then, we have

$$\begin{aligned}
I_1 &= -\int_0^b\int_{\alpha_n}^{T-\alpha_n}\varphi_1'\left(t\right)\left[\partial_{x_1}u_n\left(t,\varphi_1\left(t\right),x_2\right)\right]^2\ dt\ dx_2 \\
&= -\int_{Q_{\alpha_n}}\frac{\varphi_1'\left(t\right)}{\varphi_2\left(t\right)-\varphi_1\left(t\right)}\left(\partial_{x_1}u_n\right)^2dt\ dx_1dx_2 \\
&\quad +2\int_{Q_{\alpha_n}}\frac{\varphi_2\left(t\right)-x_1}{\varphi_2\left(t\right)-\varphi_1\left(t\right)}\varphi_1'\left(t\right)\left(\partial_{x_1}u_n\right)\left(\partial_{x_1}^2u_n\right)\ dt\ dx_1dx_2.
\end{aligned}$$

Thanks to Lemma 3.3.2, we can write

$$\int_{\varphi_1(t)}^{\varphi_2(t)}\left[\partial_{x_1}u_n\right]^2dx_1 \leq K_2\left[\varphi_2\left(t\right)-\varphi_1\left(t\right)\right]^2\int_{\varphi_1(t)}^{\varphi_2(t)}\left[\partial_{x_1}^2u_n\right]^2dx_1.$$

Therefore

$$\int_{\varphi_1(t)}^{\varphi_2(t)}\left[\partial_{x_1}u_n\right]^2\frac{|\varphi_1'|}{\varphi_2-\varphi_1}dx_1 \leq K_2^2\left|\varphi_1'\right|\left[\varphi_2-\varphi_1\right]\int_{\varphi_1(t)}^{\varphi_2(t)}\left[\partial_{x_1}^2u_n\right]^2dx_1,$$

consequently

$$\begin{aligned}
|I_1| &\leq K_2\int_{Q_{\alpha_n}}\left|\varphi_1'\right|\left[\varphi_2-\varphi_1\right]\left(\partial_{x_1}^2u_n\right)^2dt\ dx_1dx_2 \\
&\quad +2\int_{Q_{\alpha_n}}\left|\varphi_1'\right|\left|\partial_{x_1}u_n\right|\left|\partial_{x_1}^2u_n\right|\ dt\ dx_1dx_2,
\end{aligned}$$

since $\left|\dfrac{\varphi_2\left(t\right)-x_1}{\varphi_2\left(t\right)-\varphi_1\left(t\right)}\right| \leq 1$. Using the inequality

$$2\left|\varphi_1'\partial_{x_1}u_n\right|\left|\partial_{x_1}^2u_n\right| \leq \epsilon\left(\partial_{x_1}^2u_n\right)^2+\frac{1}{\epsilon}\left(\varphi_1'\right)^2\left(\partial_{x_1}u_n\right)^2$$

for all $\epsilon > 0$, we obtain

$$\begin{aligned} |I_1| &\leq K_2 \int_{Q_{\alpha_n}} |\varphi_1'| \left[\varphi_2 - \varphi_1\right] \left(\partial_{x_1}^2 u_n\right)^2 dt\ dx_1 dx_2 \\ &+ \int_{Q_{\alpha_n}} \epsilon \left(\partial_{x_1}^2 u_n\right)^2 dt\ dx_1 dx_2 + \frac{1}{\epsilon} \int_{Q_{\alpha_n}} \left(\varphi_1'\right)^2 \left(\partial_{x_1} u_n\right)^2 dt\ dx_1 dx_2. \end{aligned}$$

Lemma 3.3.2 yields

$$\frac{1}{\epsilon} \int_{Q_{\alpha_n}} \left(\varphi_1'\right)^2 \left(\partial_{x_1} u_n\right)^2 dt\ dx_1 dx_2 \leq K_2 \frac{1}{\epsilon} \int_{Q_{\alpha_n}} \left(\varphi_1'\right)^2 \left[\varphi_2 - \varphi_1\right]^2 \left(\partial_{x_1}^2 u_n\right)^2 dt\ dx_1 dx_2.$$

Thus,

$$\begin{aligned} |I_1| &\leq K_2 \int_{Q_{\alpha_n}} \left[|\varphi_1'| \left|\varphi_2 - \varphi_1\right| + \frac{1}{\epsilon} \left(\varphi_1'\right)^2 \left|\varphi_2 - \varphi_1\right|^2 \right] \left(\partial_{x_1}^2 u_n\right)^2 dt\ dx_1 dx_2 \\ &+ \int_{Q_{\alpha_n}} \epsilon \left(\partial_{x_1}^2 u_n\right)^2 dt\ dx_1 dx_2 \\ &\leq (2K_2 + 1)\, \epsilon \int_{Q_{\alpha_n}} \left(\partial_{x_1}^2 u_n\right)^2 dt\ dx_1 dx_2, \end{aligned}$$

since $|\varphi_1' \left(\varphi_2 - \varphi_1\right)| \leq \epsilon$. Finally, taking $K_4 = (2K_2^2 + 1)$, we obtain

$$|I_1| \leq K_4 \epsilon \left\|\partial_{x_1}^2 u_n\right\|_{L^2(Q_{\alpha_n})}.$$

The inequality

$$|I_2| \leq K_4 \epsilon \left\|\partial_{x_1}^2 u_n\right\|_{L^2(Q_{\alpha_n})},$$

can be proved by a similar method.

This ends the proof of Lemma 3.3.3.

2) Estimation of $-2\langle \partial_t u_n, \partial_{x_2}^2 u_n \rangle$: We have

$$\partial_t u_n \partial_{x_2}^2 u_n = \partial_{x_2} \left(\partial_t u_n \partial_{x_2} u_n\right) - \tfrac{1}{2} \partial_t \left(\partial_{x_2} u_n\right)^2.$$

Then

$$\begin{aligned} -2\langle \partial_t u_n, \partial_{x_2}^2 u_n \rangle &= -2 \int_{Q_{\alpha_n}} \partial_t u_n \partial_{x_2}^2 u_n dt\ dx_1 dx_2 \\ &= -2 \int_{Q_{\alpha_n}} \partial_{x_2} \left(\partial_t u_n \partial_{x_2} u_n\right)\ dt\ dx_1 dx_2 \\ &\quad + \int_{Q_{\alpha_n}} \partial_t \left(\partial_{x_2} u_n\right)^2\ dt\ dx_1 dx_2 \\ &= \int_{\partial Q_{\alpha_n}} \left[\left(\partial_{x_2} u_n\right)^2 \nu_t - 2 \partial_t u_n \partial_{x_2} u_n \nu_{x_2} \right] d\sigma. \end{aligned}$$

Using the Cauchy-Dirichlet boundary conditions, we see that the above boundary integral is nonnegative. Consequently

$$-2\left\langle \partial_t u_n, \partial_{x_2}^2 u_n \right\rangle \geq 0. \tag{3.3.2}$$

3) Estimation of $2\langle \partial_{x_1}^2 u_n, \partial_{x_2}^2 u_n \rangle$: We have

$$\partial_{x_1}^2 u_n.\partial_{x_2}^2 u_n = \partial_{x_1}\left(\partial_{x_1} u_n.\partial_{x_2}^2 u_n\right) - \partial_{x_2}\left(\partial_{x_1} u_n.\partial_{x_1}\partial_{x_2} u_n\right) + \left(\partial_{x_1}\partial_{x_2} u_n\right)^2.$$

Then

$$\begin{aligned}
2\langle \partial_{x_1}^2 u_n, \partial_{x_2}^2 u_n \rangle &= 2\int_{Q_{\alpha_n}} \partial_{x_1}^2 u_n.\partial_{x_2}^2 u_n dt\ dx_1 dx_2 \\
&= 2\int_{Q_{\alpha_n}} \partial_{x_1}\left(\partial_{x_1} u_n.\partial_{x_2}^2 u_n\right) dt\ dx_1 dx_2 \\
&\quad -2\int_{Q_{\alpha_n}} \partial_{x_2}\left(\partial_{x_1} u_n.\partial_{x_1}\partial_{x_2} u_n\right) dt\ dx_1 dx_2 \\
&\quad +2\int_{Q_{\alpha_n}} \left(\partial_{x_1}\partial_{x_2} u_n\right)^2 dt\ dx_1 dx_2 \\
&= 2\int_{Q_{\alpha_n}} \left(\partial_{x_1}\partial_{x_2} u_n\right)^2 dt\ dx_1 dx_2 \\
&\quad +2\int_{\partial Q_{\alpha_n}} \left[\partial_{x_1} u_n \partial_{x_2}^2 u_n \nu_{x_1} - \partial_{x_1} u_n.\partial_{x_1}\partial_{x_2} u_n \nu_{x_2}\right] d\sigma.
\end{aligned}$$

Thanks to the boundary conditions, we obtain

$$2\langle \partial_{x_1}^2 u_n, \partial_{x_2}^2 u_n \rangle \geq 2\left\|\partial_{x_1}\partial_{x_2} u_n\right\|_{L^2(Q_{\alpha_n})}^2. \tag{3.3.3}$$

Then, summing up the estimates (3.3.1), (3.3.2) and (3.3.3) of the inner products, and making use of Lemma 3.3.3, we then obtain

$$\begin{aligned}
\|f_n\|_{L^2(Q_{\alpha_n})}^2 &\geq \|\partial_t u_n\|_{L^2(Q_{\alpha_n})}^2 + \left\|\partial_{x_1}^2 u_n\right\|_{L^2(Q_{\alpha_n})}^2 + \left\|\partial_{x_2}^2 u_n\right\|_{L^2(Q_{\alpha_n})}^2 \\
&\quad - |I_1| - |I_2| + 2\left\|\partial_{x_1}\partial_{x_2} u_n\right\|_{L^2(Q_{\alpha_n})}^2 \\
&\geq \|\partial_t u_n\|_{L^2(Q_{\alpha_n})}^2 + (1 - 2K_4\epsilon)\left\|\partial_{x_1}^2 u_n\right\|_{L^2(Q_{\alpha_n})}^2 \\
&\quad + \left\|\partial_{x_2}^2 u_n\right\|_{L^2(Q_{\alpha_n})}^2 + 2\left\|\partial_{x_1}\partial_{x_2} u_n\right\|_{L^2(Q_{\alpha_n})}^2.
\end{aligned}$$

Then, it is sufficient to choose ϵ such that $(1 - 2K_4\epsilon) > 0$ to get a constant $K_0 > 0$ independent of n such that

$$\|f_n\|_{L^2(Q_{\alpha_n})} \geq K_0 \|u_n\|_{H^{1,2}(Q_{\alpha_n})},$$

and since

$$\|f_n\|_{L^2(Q_{\alpha_n})} \;\leq\; \|f\|_{L^2(Q)},$$

there exists a constant $K_1 > 0$, independent of n satisfying

$$\|u_n\|_{H^{1,2}(Q_{\alpha_n})} \leq K_1 \|f_n\|_{L^2(Q_{\alpha_n})} \leq K_1 \|f\|_{L^2(Q)}.$$

This completes the proof of Proposition 3.3.1. ■

3.4 Passage to the limit

We are now able to prove the main result of this work

Theorem 3.4.1 *We assume that φ_1 and φ_2 fulfil the conditions (3.1.2) and (3.1.3), then the heat operator*

$$L = \partial_t - \partial_{x_1}^2 - \partial_{x_2}^2$$

is an isomorphism from $H_0^{1,2}(Q)$ into $L^2(Q)$.

Proof. Choose a sequence Q_{α_n} $n = 1, 2, \ldots$ of reference domains (see Section 3.2) such that $Q_{\alpha_n} \subseteq Q$ with (α_n) a sequence decreasing to 0, as $n \to \infty$. Then we have $Q_{\alpha_n} \to Q$, as $n \to \infty$.

Consider the solution $u_{\alpha_n} \in H^{1,2}(Q_{\alpha_n})$ of the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u_{\alpha_n} - \partial_{x_1}^2 u_{\alpha_n} - \partial_{x_2}^2 u_{\alpha_n} = f & \text{in } Q_{\alpha_n} \\ \\ u_{\alpha_n / \partial Q - \Gamma_{T-\alpha_n}} = 0, \end{cases}$$

with $\Gamma_{T-\alpha_n}$ is the part of the boundary of Q_{α_n} where $t = T - \alpha_n$. Such a solution u_{α_n} exists by Theorem 3.2.1. Let $\widetilde{u_{\alpha_n}}$ the 0-extension of u_{α_n} to Q. In virtue of Proposition 3.3.1, we know that there exists a constant C such that

$$\|\widetilde{u_{\alpha_n}}\|_{L^2(Q)} + \left\|\widetilde{\partial_t u_{\alpha_n}}\right\|_{L^2(Q)} + \sum_{\substack{i,j=0 \\ 1 \leq i+j \leq 2}}^{2} \left\|\widetilde{\partial_{x_1}^j \partial_{x_2}^j u_{\alpha_n}}\right\|_{L^2(Q)} \leq C \|f\|_{L^2(Q)}.$$

This means that $\widetilde{u_{\alpha_n}}$, $\widetilde{\partial_t u_{\alpha_n}}$, $\widetilde{\partial_{x_1}^j \partial_{x_2}^j u_{\alpha_n}}$ for $1 \leq i+j \leq 2$ are bounded functions in $L^2(Q)$. So for a suitable increasing sequence of integers n_k, $k = 1, 2, \ldots$, there exist functions

$$u,\ v \text{ and } v_{i,j}\ 1 \leq i+j \leq 2$$

in $L^2(Q)$ such that

$$\begin{array}{lcll} \widetilde{u_{\alpha_{n_k}}} & \rightharpoonup & u & \text{weakly in } L^2(Q), \quad k \to \infty \\ \widetilde{\partial_t u_{\alpha_{n_k}}} & \rightharpoonup & v & \text{weakly in } L^2(Q), \quad k \to \infty \\ \widetilde{\partial_{x_1}^j \partial_{x_2}^j u_{\alpha_n}} & \rightharpoonup & v_{i,j} & \text{weakly in } L^2(Q), \quad k \to \infty, 1 \leq i+j \leq 2. \end{array}$$

Let then $\theta \in D(Q)$. For n_k large enough we have supp $\theta \subset Q_{\alpha_{n_k}}$. Thus

$$\begin{aligned} \langle v_{1,0}, \theta \rangle_{D'(Q)\times D(Q)} &= \lim_{n_k \longrightarrow \infty} \int_Q \widetilde{\partial_{x_1} u_{\alpha_{n_k}}}.\theta \; dt dx_1 dx_2 \\ &= \lim_{n_k \longrightarrow \infty} \int_{Q_{\alpha_{n_k}}} \partial_{x_1} u_{\alpha_{n_k}}.\theta \; dt dx_1 dx_2 \\ &= \lim_{n_k \longrightarrow \infty} \left\langle \partial_{x_1} u_{\alpha_{n_k}}, \theta \right\rangle_{D'\left(Q_{\alpha_{n_k}}\right)\times D\left(Q_{\alpha_{n_k}}\right)} \\ &= -\lim_{n_k \longrightarrow \infty} \left\langle u_{\alpha_{n_k}}, \partial_{x_1}\theta \right\rangle_{D'\left(Q_{\alpha_{n_k}}\right)\times D\left(Q_{\alpha_{n_k}}\right)} \\ &= -\lim_{n_k \longrightarrow \infty} \int_Q \widetilde{u_{\alpha_{n_k}}}.\partial_{x_1}\theta \; dt dx_1 dx_2 \\ &= -\lim_{n_k \longrightarrow \infty} \left\langle \widetilde{u_{\alpha_{n_k}}}, \partial_{x_1}\theta \right\rangle_{D'(Q)\times D(Q)} \\ &= -\langle u, \partial_{x_1}\theta \rangle_{D'(Q)\times D(Q)} \\ &= \langle \partial_{x_1} u, \theta \rangle_{D'(Q)\times D(Q)}. \end{aligned}$$

Then, $v_{1,0} = \partial_{x_1} u$ in $D'(Q)$ and so in $L^2(Q)$. By a similar manner, we prove that

$$v = \partial_t u,\ v_{i,j} = \partial_{x_1}^i \partial_{x_2}^j u,\ 1 \leq i+j \leq 2$$

in the sense of distributions in Q and so in $L^2(Q)$. Finally, $u \in H^{1,2}(Q)$. On the other hand,

$$\partial_t u_{\alpha_{n_k}} - \partial_{x_1}^2 u_{\alpha_{n_k}} - \partial_{x_2}^2 u_{\alpha_{n_k}} = f_{n_k} = f_{/Q_{\alpha_{n_k}}}$$

and

$$\widetilde{\partial_t u_{\alpha_{n_k}}} - \widetilde{\partial_{x_1}^2 u_{\alpha_{n_k}}} - \widetilde{\partial_{x_2}^2 u_{\alpha_{n_k}}} = \widetilde{f_{n_k}}.$$

But

$$\widetilde{f_{n_k}} \longrightarrow f \text{ in } L^2(Q)$$

and

$$\widetilde{\partial_t u_{\alpha_{n_k}}} - \widetilde{\partial^2_{x_1} u_{\alpha_{n_k}}} - \widetilde{\partial^2_{x_2} u_{\alpha_{n_k}}} \rightharpoonup \partial_t u - \partial^2_{x_1} u - \partial^2_{x_2} u.$$

So, we have

$$\partial_t u - \partial^2_{x_1} u - \partial^2_{x_2} u = f \text{ in } Q$$

On the hand, the solution u satisfies the boundary conditions $u_{/\partial Q - \Gamma_T} = 0$ since

$$\forall n \in \mathbb{N}, u_{/Q_{\alpha_n}} = u_{\alpha_n}.$$

This proves the existence of a solution to Problem 3.1.1.

Notice that we have the estimate

$$\|u\|^2_{H^{1.2}(Q)} \leq K \|f\|^2_{L^2(Q)},$$

which implies the uniqueness of the solution. ■

Remark 3.4.1 *The result given in Theorem 3.4.1 holds true only under the assumption (3.1.2) (respectively, (3.1.3)), if $\varphi_1(0) = \varphi_2(0)$ and $\varphi_1(T) < \varphi_2(T)$ (respectively, if $\varphi_1(0) < \varphi_2(0)$ and $\varphi_1(T) = \varphi_2(T)$).*

Remark 3.4.2 *Note that this work may be extended at least in the following directions:*

1. The non-regular domain Q may be replaced by a non-cylindrical domain (conical domain, for example).

2. The function f on the right-hand side of the equation of Problem (3.1.1), may be taken in $L^p(Q)$, where $p \in]1, \infty[$. The method used here does not seem to be appropriate for the space $L^p(Q)$ when $p \neq 2$.

3. The operator L may be replaced by a high order operator.