CHAPTER 2

Preliminaries

In this chapter, we collect some basic definitions and results that we need to develop further arguments in the following chapters.

2.1 Function spaces

2.1.1 Anisotropic Sobolev spaces

We introduce the so-called anisotropic Sobolev spaces $H^{r,s}$ built on the Lebesgue space of square integrable functions L^2. These function spaces are the naturel ones adopted in the study of parabolic equations and are different from those in the study of elliptic equations since the space variable x and time variable t play different roles in parabolic equations.

We recall the following definition of anisotropic Sobolev spaces (see [43])

$$H^{r,s}\left(\mathbb{R}^2\right)=\left\{u\in L^2\left(\mathbb{R}^2\right):\left[\left(1+\zeta^2\right)^{r/2}+\left(1+\tau^2\right)^{s/2}\right]\widehat{u}\in L^2\left(\mathbb{R}^2\right)\right\}$$

where $\widehat{u}$ is the Fourier transform of u and r, s are two non-negative numbers. We put

$$H^{r,s}\left(\Omega\right)=\left\{u_{/\Omega}:u\in H^{r,s}\left(\mathbb{R}^2\right)\right\}, \tag{2.1.1}$$

with Ω is an open subset of $\mathbb{R}^2$.

Now, we give some basic properties of the anisotropic Sobolev space $H^{1,2}$.

The following result for the symmetric Sobolev space H^1, may be extended to anisotropic Sobolev space $H^{1,2}$

Theorem 2.1.1 *[19]. Let Q be a bounded open set with Lipschitz boundary and Q_1, Q_2 two open subsets of Q with Lipschitz boundaries such that*

$$\overline{Q_1} \cup \overline{Q_2} = \overline{Q},$$
$$Q_1 \cap Q_2 = \emptyset.$$

Set $\Gamma = \partial Q_1 \cap \partial Q_2$. Let $u_1 \in H^1(Q_1)$, $u_2 \in H^1(Q_2)$ satisfying

$$u_1 = u_2 \text{ on } \Gamma,$$

then the function u defined by

$$u = \begin{cases} u_1 \text{ in } Q_1 \\ u_2 \text{ in } Q_2, \end{cases}$$

belongs to $H^1(Q)$.

Proof. It is clear that $u \in L^2(Q)$. For an arbitrary $i \in \{1, 2, ..., n\}$ and a fixed $\varphi \in D(Q)$, we have

$$\begin{aligned} \left\langle \frac{\partial u}{\partial x_i}, \varphi \right\rangle &= -\left\langle \frac{\partial \varphi}{\partial x_i}, u \right\rangle \\ &= -\int_{Q_1} u_1 \frac{\partial \varphi}{\partial x_i} dx - \int_{Q_2} u_2 \frac{\partial \varphi}{\partial x_i} dx. \end{aligned}$$

For $k = 1, 2$

$$\int_{Q_k} u_k \frac{\partial \varphi}{\partial x_i} dx = -\int_{Q_k} \frac{\partial u_k}{\partial x_i} \varphi dx + \int_{\Gamma} u_k \varphi \nu_i^{(k)} dx$$

because φ vanishes on $\partial Q_k \backslash \Gamma$, here $\nu^{(k)}$ is the outward normal vector on ∂Q_k. So, since $\nu^{(2)} = -\nu^{(1)}$ on Γ, we have

$$\begin{aligned} \left\langle \frac{\partial u}{\partial x_i}, \varphi \right\rangle &= \int_{Q_1} \frac{\partial u_1}{\partial x_i} \varphi dx - \int_{Q_2} \frac{\partial u_2}{\partial x_i} \varphi dx \\ &\quad -\int_{\Gamma} (u_1 - u_2) \varphi \nu_i^{(1)} dx. \end{aligned}$$

The boundary integral vanish, so we obtain

$$\frac{\partial u}{\partial x_i} = \begin{cases} \dfrac{\partial u_1}{\partial x_i} \text{ in } Q_1 \\ \dfrac{\partial u_2}{\partial x_i} \text{ in } Q_2. \end{cases}$$

Since each $\dfrac{\partial u_k}{\partial x_i}$ belongs to $L^2(Q_k)$, we conclude that $\dfrac{\partial u}{\partial x_i} \in L^2(Q)$. ■

Lemma 2.1.1 *If* $u \in H^{1,2}\left(]0,T[\times]0,1[\right)$, *then* $u_{/t=0} \in H^1(\gamma_0)$, $u_{/x=0} \in H^{\frac{3}{4}}(\gamma_1)$ *and* $u_{/x=1} \in H^{\frac{3}{4}}(\gamma_2)$, *where* $\gamma_0 = \{0\}\times]0,1[$, $\gamma_1 =]0,T[\times\{0\}$ *and* $\gamma_2 =]0,T[\times\{1\}$.

The above lemma is a particular case of [43, Vol. 2, Theorem 2.1].

Lemma 2.1.2 *Let* T *and* b *two positive numbers. Then, the space*

$$D\left(]0,T[\,;H^4\left(]0,1[\times]0,b[\right)\cap H_0^1\left(]0,1[\times]0,b[\right)\right),$$

(see [43, Vol. 2, p.13]) is dense in the subspace of $H^{1,2}\left(]0,T[\times]0,1[\times]0,b[\right)$ *defined by*

$$u = 0 \ \text{on} \ ]0,T[\times\{0\}\times]0,b[\ \text{and} \ ]0,T[\times\{1\}\times]0,b[\,.$$

It is a particular case of [43, Vol. 1, Theorem 2.1].

2.1.2 Interpolation spaces

In this subsection we only recall concepts of interpolation theory that are needed for our purposes.

Let X, Y be two Hilbert spaces with

$$X \subset Y \text{ continuously.}$$

There are various equivalent methods that allow us to build spaces

$$[X,Y]_\theta \ \ 0<\theta<1,$$

"intermediate" between X and Y. We give here one of the usual methods, namely, that of Lions-Peetre [42].

Definition 2.1.1 *The space* $[X,Y]_\theta$ $0<\theta<1$, *is a sub-space of* Y *consisting of elements* a *which can be written in the form*

$$a = \int_0^\infty u(t)\,\frac{dt}{t} \tag{2.1.2}$$

with

$$t^\theta u(t) \in L_*^2(X),\ t^{\theta-1}u(t) \in L_*^2(Y). \tag{2.1.3}$$

This space is endowed with the norm

$$a \longmapsto \inf \left[\left(\int_0^\infty t^{2\theta} \left| u(t) \right|_X^2 \frac{dt}{t} \right)^{\frac{1}{2}} + \left(\int_0^\infty t^{2(\theta-1)} \left| u(t) \right|_Y^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right];$$

the inf *is taken with respect to* u *verifying 2.1.2 and 2.1.3.*

Here L_*^2 *denotes the space of square integrable functions* $f : (0, +\infty) \to V$ *with the Haar measure* dt/t.

Example 2.1.1 $H^{r,s}(\Omega)$ *can also be defined as a real interpolation space between* $H^{r/(1-\theta),s/(1-\theta)}(\Omega)$ *and* $L^2(\Omega)$, $\theta \in]0,1[$, *(see [67])*

$$H^{r,s}(\Omega) = \left[H^{r/(1-\theta),s/(1-\theta)}(\Omega), L^2(\Omega) \right]_\theta. \tag{2.1.4}$$

In this work, we consider the case $s = 2r$, $\theta = 1 - r$,

$$H^{r,2r}(\Omega) = \left[H^{1,2}(\Omega), L^2(\Omega) \right]_{1-r} \quad \forall r \in]0,1[. \tag{2.1.5}$$

The main features of the spaces $H^{r,2r}$ available to second-order parabolic equations is that smoothness with respect to spatial variables is twice as high with respect to time. In other words, the weak derivatives with respect to t does not exceed half of the highest order of weak derivatives with respect to x. The space $H^{r,2r}(\Omega)$ is well defined by Relationship (2.1.5) because the right hand side term of (2.1.5) is well defined as an interpolation space between two well defined spaces $H^{1,2}(\Omega)$ and $L^2(\Omega)$.

Putting $s = 2r$ in Relationship (2.1.1), we obtain

$$H^{r,2r}(\Omega) = \left\{ u_{/\Omega} : u \in H^{r,2r}\left(\mathbb{R}^2\right) \right\}. \tag{2.1.6}$$

This relation also give a complete definition of the space $H^{r,2r}(\Omega)$, so it is important to know if the space given by Relationship (2.1.5) is the same as that given by Relationship (2.1.6). The answer to this question depends on the geometry of Ω.

If Ω has the continuation property with respect to the space $H^{1,2}(\Omega)$, then

$$\left\{ u_{/\Omega} : u \in H^{r,2r}\left(\mathbb{R}^2\right) \right\} = \left[H^{1,2}(\Omega), L^2(\Omega) \right]_{1-r} \quad \forall r \in]0,1[.$$

If Ω has not this property, then the spaces may be different. However, we have

$$\left\{ u_{/\Omega} : u \in H^{r,2r}\left(\mathbb{R}^2\right) \right\} \subset \left[H^{1,2}(\Omega), L^2(\Omega) \right]_{1-r} \quad \forall r \in]0,1[,$$

see [56]. The continuation property is called "the horn property" of Besov [9], and it corresponds to the cone property for the symmetric Sobolev spaces.

Definition 2.1.2 *Let Ω be an open subset of $\mathbb{R}^n$. Ω is said to have the "horn property" of Besov if each of their points can be reached from within as the vertex of some horn. By a "horn" is meant a set*

$$R = \left\{ \alpha_i h < x_i^{k_i} < \beta_i h; i = 1, 2, ..., n, 0 < h < 1 \right\}$$

$0 < \alpha_i < \beta_i < \infty, 0 < k_i < \infty, i = 1, 2, ..., n,$ *see Fig. 4.*

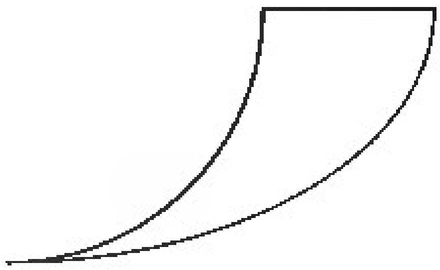

Fig. 4: A horn in the plane.

Now, we present some results interpolation spaces.

Theorem 2.1.2 *[67]. Let A_0, A_1 be two Hilbert spaces with*

$$A_0 \subset A_1 \text{ continuously.}$$

Then

$$\left[L^2(A_0), L^2(A_1) \right]_\theta = L^2([A_0, A_1]_\theta),$$

$0 < \theta < 1.$

A direct consequence of Theorem 2.1.2 is

Corollary 2.1.1 *For each $0 \leq r \leq 1$*

$$\begin{aligned} \left[L^2(\mathbb{R}_+; H^2(\Omega)), L^2(\mathbb{R}_+; H^1(\Omega)) \right]_{1-r} &= L^2\left(\mathbb{R}_+; \left[H^2(\Omega), H^1(\Omega) \right]_{1-r} \right) \\ &= L^2(\mathbb{R}_+; H^{1+r}(\Omega)), \end{aligned}$$

with Ω an open bounded set of $\mathbb{R}^n$.

2.1.3 The spaces $H^{\frac{1}{2}}, H_0^{\frac{1}{2}}$ and $H_{00}^{\frac{1}{2}}$

We will need in Chapter 5 some particular Sobolev spaces, namely, $H_0^r(\Omega)$, $0 \leq r \leq 1$.

We recall the following definition of the Sobolev spaces $H^{\frac{1}{2}}(\mathbb{R})$ (see [43])

$$H^{\frac{1}{2}}(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) : \sqrt{|x|}\widehat{u} \in L^2(\mathbb{R}) \right\},$$

where $\widehat{u}$ is the Fourier transform of u.

Hereafter a caracterization of $H^{\frac{1}{2}}(\mathbb{R})$.

Theorem 2.1.3

$$u \in H^{\frac{1}{2}}(\mathbb{R}) \Longrightarrow \frac{u(x) - u(-x)}{\sqrt{|x|}} \in L^2(\mathbb{R}).$$

Remark 2.1.1 *Let Ω a bounded set of $\mathbb{R}^n$ with Lipschitz boundary, then for every $\theta \in]0, 1[$:*

$$H_0^{\theta}(\Omega) = \left[H_0^1(\Omega), L^2(\Omega) \right]_{1-\theta}$$

except for $\theta = \frac{1}{2}$.

$$H_{00}^{\frac{1}{2}}(\Omega) :\equiv \left(H_0^1(\Omega), L^2(\Omega) \right)_{\frac{1}{2}} \neq H^{\frac{1}{2}}(\Omega)$$

for $\theta = \frac{1}{2}$.

$$H_0^{\theta}(\Omega) = H^{\theta}(\Omega)$$

for $0 \leq \theta < \frac{1}{2}$.

Hereafter a caracterization of $H_{00}^{\frac{1}{2}}(\Omega)$ that we can find in [43].

Theorem 2.1.4

$$H_{00}^{\frac{1}{2}}(\Omega) = \left\{ u \in H^{\frac{1}{2}}(\Omega) : \widetilde{u} \in H^{\frac{1}{2}}(\mathbb{R}^n) \right\}$$
$$= \left\{ u \in H^{\frac{1}{2}}(\Omega) : \frac{u}{\sqrt{d(x)}} \in L^2(\Omega) \right\}$$

where $\widetilde{u}$ is the 0-extension of u and $d(x)$ is the distance of x to the origin.

2.2 Results on some parabolic model problems

Many results of the theory of parabolic equations for smooth domains are not true if the boundary of the domain is not regular. On the other hand, the methods which were developed for domains with smooth boundaries cannot be directly applied to domains with irregularities. In this section, we give results on some parabolic model problems, both in the regular and irregular cases.

Problem 2.2.1 *Consider the following parabolic boundary problem*

$$\begin{cases} \partial_t u - a\ \partial_x^2 u - b\ \partial_y^2 u - 2c\ \partial_x \partial_y u = f, & (t,x,y) \in \left]0,T\right[\times \Omega \\ u(0,x) = 0, & (x,y) \in \Omega \\ u(t,\sigma) = 0, & (t,\sigma) \in \left]0,T\right[\times \Gamma \end{cases} \tag{2.2.1}$$

where T is a finite positive number and Ω is any rectangle of $\mathbb{R}^2$. Here $f \in L^2(Q)$ and the coefficients a, b and c real-valued functions defined on $[0,T]$, Lipschitz continuous on $]0,T[$.

We have the following result

Theorem 2.2.1 *[11, Theorem 7.22]. For each $f \in L^2(Q)$, there exists a unique solution $u \in H^{1,2}(Q)$ of Problem (2.2.1).*

Proof. Set $X = L^2(0,T)$ and $u(t) = u(t,.,.)$, then Problem (2.2.1) is equivalent to the following abstract Cauchy problem in X

$$\begin{cases} u'(t) + L(t)\,u(t) = f(t), & t \in (0,T), \\ u(0) = 0, \end{cases} \tag{2.2.2}$$

where the family $(L(t))_{t\in[0,T]}$ is defined by

$$\begin{aligned} D(L(t)) &= H^2(\Omega) \cap H_0^1(\Omega), \\ (L(t)\psi)(x,y) &= -a\ \partial_x^2\psi - b\ \partial_y^2\psi - 2c\ \partial_x\partial_y\psi \text{ for a.e. } t \in (0,T). \end{aligned}$$

Observe that $\overline{D(L(t))} = X$. We obtain then the new abstract form of the previous problem, mainly

$$Au + Bu = f,$$

where

$$\begin{aligned} D(A) &= \{u \in L^2(Q) : u' \in L^2(Q) \text{ and } u(0) = 0\} \\ (Au)(t) &= u'(t), \, t \in [0,1]. \end{aligned}$$

and

$$\begin{aligned} D(B) &= \{u \in L^2(Q) : u \in D(L(t)), \text{ a.e. } t \in (0,T)\} \\ (Bu)(t) &= L(t)\, u(t), \, t \in [0,T], \end{aligned}$$

Now we are in position to apply the result of the sums of operators, see Chapter 1, subsection 1.3.2. For this purpose we must verify the assumptions of Theorem 1.3.2. The spectral properties of A and B are as follows. ■

Proposition 2.2.1 *A and B are linear closed operators and their domains are dense in $L^2(Q)$. Moreover, they satisfy Assumptions (1.3.5), (1.3.6) and (1.3.7).*

Proof. The proof of this result can be found in [11, Theorem 7.22]. ■

In addition A and B satisfy Labbas-Terreni condition (1.3.8) with $\tau = \rho = 0$. This ends the proof of Theorem (2.2.1).

Remark 2.2.1 *In the case of a regular domain Ω, the corresponding result can be found in [43, Vol.2].*

Problem 2.2.2 *Let Q be the rectangle $]0,T[\times]0,1[$, $f \in L^2(Q)$ and $\psi \in H^1(\gamma_0)$. Consider the following boundary value problem*

$$\begin{cases} \partial_t u - \partial_x^2 u = f \text{ in } Q \\ u_{/\gamma_0} = \psi \\ \partial_x u + \beta_i(t)\, u_{/\gamma_i} = 0, \; i = 1, \, 2, \end{cases} \tag{2.2.3}$$

where $\gamma_0 = \{0\} \times]0,1[$, $\gamma_1 =]0,T[\times \{0\}$ and $\gamma_2 =]0,T[\times \{1\}$. The coefficients $\beta_i(t)$ are smooth functions.

Proposition 2.2.2 *[43, Theorem 4.3, Vol. 2]. Problem (2.2.3) admits a (unique) solution $u \in H^{1,2}(Q)$.*

Remark 2.2.2 *In the application of [43, Theorem 4.3, Vol.2], we can observe that there are no compatibility conditions to satisfy because $\partial_x \psi$ is only in $L^2(\gamma_0)$.*

Problem 2.2.3 *Let Ω_0 be an open bounded set of $\mathbb{R}^n$ with boundary Γ and Q_0 the cylinder $\mathbb{R}_+ \times \Omega_0$ with lateral boundary $\Sigma = \mathbb{R}_+ \times \Gamma$. We assume that Ω_0 is convex or of class C^2.*

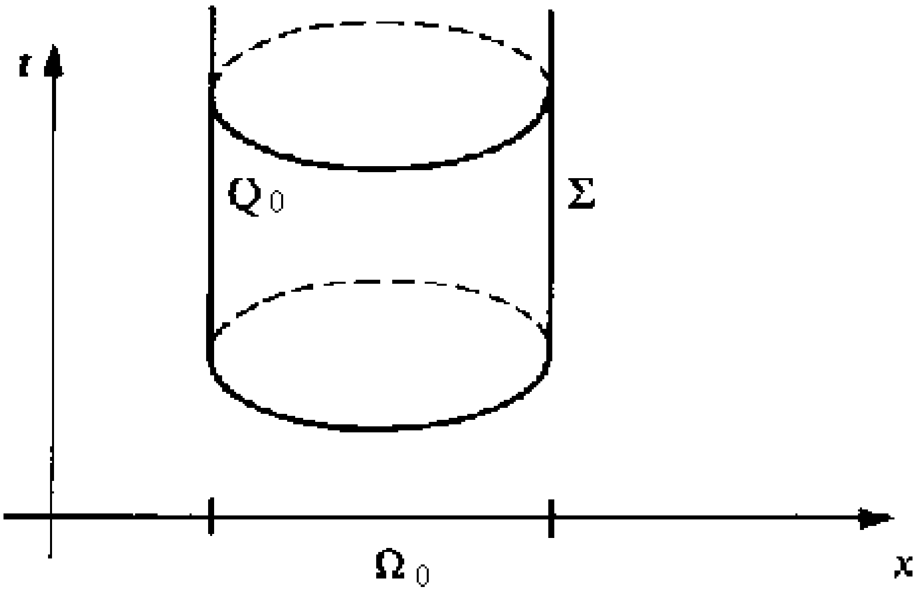

Fig. 5 :The cylinder $Q_0 = R_+ \times \Omega_0$

Consider in Q_0 the following boundary value problem

$$\begin{cases} \partial_t u - \Delta u = 0 \quad in\ \mathbb{R}_+ \times \Omega_0 \\ u = 0 \quad on\ \Sigma \\ u(0,x) = u_0(x),\ x \in \Omega_0. \end{cases} \tag{2.2.4}$$

We will need the following well known result (cf. Lions and Magenes [43] and Brezis [10]) which gives the regularity of the solution u of (2.2.4) in terms of the regularity of the initial data u_0.

Theorem 2.2.2 *1) For given u_0 in $H_0^1(\Omega_0)$, Problem (2.2.4) has a unique solution u in $H^{1,2}(Q_0)$ defined by $H^{1,2}(Q_0) = L^2(\mathbb{R}_+; H^2(\Omega_0)) \cap H^1(\mathbb{R}_+; L^2(\Omega_0))$. Moreover,*

$$\int_0^T \|\partial_t u\|^2_{L^2(\Omega_0)}\, dt + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega_0)} = \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_0)} \quad \forall T > 0.$$

2) For given u_0 in $L^2(\Omega_0)$, Problem (2.2.4) has a unique weak solution

$$u \in L^2\left(\mathbb{R}_+; H_0^1(\Omega_0)\right) \cap H^1\left(\mathbb{R}_+; H^{-1}(\Omega_0)\right) \cap L^\infty\left(\mathbb{R}_+; L^2(\Omega_0)\right).$$

Moreover

$$\frac{1}{2}\|u\|^2_{L^2(\Omega_0)} + \int_0^T \|\nabla u\|^2_{L^2(\Omega_0)}\,dt = \frac{1}{2}\|u_0\|^2_{L^2(\Omega_0)} \quad \forall T > 0.$$