CHAPTER 1 Parabolic problems in non-cylindrical domains

1.1 Historical notes

The history of boundary value problems for parabolic equations in non-cylindrical domains starts at the beginning of the 20th century with the pioneering work of Gevrey [18], in which, existence results for second order parabolic equations for a sufficiently small interval of time t have been established.

The Gevrey-type results were then followed by I. G. Petrovskii [51] in 1934 and A. N. Tikhonov [66] in 1937 and others from the beginning of the fifties until now.

I. G. Petrovskii was the first to study parabolic equations in non-cylindrical domains with characteristic points. He studied in [52] the question of regularity of boundary points for a plan domain Q bounded by the lines $t = 0$ and $t = T$ and the curves $x = \psi_1(t)$ and $x = \psi_2(t)$, with $\psi_1(t) \leq \psi_2(t)$ for the equation of the heat conduction and he found necessary and sufficient conditions on the tangency order of the boundary hypersurface with a plane $t = \text{const}$ for solvability of the Dirichlet problem for a such equation. Petrovskii constructed also examples to show that a point of the boundary can be regular for the equation $u_t = u_{xx}$ and non-regular for the equation $u_t = a^2 u_{xx}$, where a is a positive constant such that $a \neq 1$. These results have important applications in probability theory.

Later on, these results were generalized by Landis [38] and Evans and Gariepy [15] to the case of domains $Q \subset \mathbb{R}^n_{t,x}$ with boundaries of arbitrary structures. In [66] Tikhonov studied the equation of heat conduction in a cylindrical domain $G = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n_x$ and $0 < t < T$. He showed that a point $(x_0, t_0) \in \partial\Omega \times [0,T]$ is regular for this equation if and only if the point x_0 is regular for the Laplace equation in Ω. The regularity of the points of the boundary for second order parabolic equations under various assumptions on their coefficients is studied in [53], [14], [36], [37], and [45]. The same question for second order degenerate parabolic equations, is considered in [49] and [50] where an extensive list of references can also be found.

At the beginnig of the fifties, the study of parabolic equations in non-cylindrical domains was initiated by Fichera [16], who considered a class of second order equations, then followed by Lions [41], who proved existence and uniqueness of weak solutions for a large class of higher order equations and systems, but his class of domains is much smaller, as he only considers domains which can be written locally as the graph of a C^∞ function. The solvability of boundary-value problems for parabolic equations in non-smooth domains has been the subject of systematic studies since the 1960s (see [28] and the references therein). In the late 1950s and early 1960 Friedmann [17], Solonnikov [65] and others obtained *a priori* estimates for the heat equation, and for much general parabolic equations and systems in bounded non-cylindrical domains in higher dimensions. These results are analogous to Schauder's theorems on *a priori* estimates for solutions of linear elliptic equations. The solvability of the first boundary-value problem for higher-order parabolic equations in Sobolev spaces was studied by Mikhailov [47]. The case of general boundary-value problems in non-cylindrical domains of special form was considered by Kondrat'ev [27], where asymptotic expansion of the solutions in the neighbourhood of non-regular points were obtained. Boundary value problems for second order parabolic equations in angular domains or in domains with corners or edges are studied in [63], [39] [6] and [7]. Since the beginning of the seventies, many classical methods have been successfully applied for solving parabolic problems in non-cylindrical domains and we can recognize at least five powerful methods: Elliptic regularization method, domain decomposition method, layer potential method, sum of operators method and Rothe's method. Equations that are 2m-parabolic in a plane non-smooth domain are considered by Baderko [8] and by Sadallah in [57], [58], who obtained some results on conditions for the smoothness of the solution near singular points of the boundary by using the domain decomposition method. This method is based on the approximation of the non-smooth domain by a sequence of sub-domains which can be transformed into smooth domains.

The number of papers devoted to parabolic equations in non-cylindrical domains increased considerably from 90's. This was largely due to the discovery of new applications and to the successful adaptation of some classical methods to solve them. One of the powerful methods which has been successfully applied for solving parabolic problems in non-cylindrical domains is the operators sum method. Let us mention, for example, the works [32], [33], [31], by R. Labbas, A. Medeghri and B.-K. Sadallah where some fainly new results can be found. The method of layer potential is used by Hofmann and Lewis [22] for the solvability of the heat equation in non-cylindrical domains satisfying some conditions of Lipschitz's type. Rothe's classical method was extended so that it can be used to solve some linear parabolic boundary value problems in non-cylindrical domains, see [12] and [29]. Yu. A. Alkhutov considered in [2] the Dirichlet problem for the heat equation in bounded and unbounded domains of paraboloid type with isolated characteristic points at the boundary. He found necessary and sufficient conditions in terms of the weight ensuring the unique solubility of this problem in weighted Sobolev L_p-spaces. In particular, he established in [3] a criterion for the solubility of the problem in the classical Sobolev spaces in the case when the domain is a ball.

We do not claim that our survey is exhaustive, since many problems of mechanics and physics lead directly to parabolic problems in non-smooth domains and a great many papers have been devoted to the study of parabolic equations in specific non-cylindrical domains. A systematic presentation of some questions of the theory of parabolic equations in non-smooth domains can be found in [28] and [40].

1.2 Physical applications

Many important applied problems reduce to the study of boundary-value problems for partial differential equations in domains with non-regular points on the boundary. Such questions have been discussed extensively in the literature. One of the most important equations of mathematical physics is the heat equation. This equation in two independent variables arises in problems of the plane theory of evolution equations. General second-order parabolic equations set in $U_T = U \times (0, T)$ for some fixed time $T > 0$, describe in physical applications the time-evolution of the density of some quantity u, say a chemical concentration, within the region U.

In this section, we give some physical applications for the heat equation in the case of a time-varying domain.

Example 1.2.1 *Consider the following diffusion equation in one space dimension*

$$\begin{cases} \partial_t u - \partial_x^2 u = f \in L^p(\Omega), \\ u_{/\partial_p \Omega} = 0. \end{cases} \tag{1.2.1}$$

The problem is set in a non-rectangular domain. More precisely, the standard domain we consider is the curvilinear triangle Ω :

$$\Omega = \left\{(t, x) \in \mathbb{R}^2 : t \in \left]0, 1\right[, 0 < x < t^\alpha = \varphi(t)\right\}, \alpha \geq 1/2.$$

The parabolic boundary $\partial_p \Omega$ *of* Ω *is defined by*

$$\partial_p \Omega = (\Gamma_1 \cup \Gamma_3) \setminus \Gamma_2$$

with

$$\begin{aligned} \Gamma_1 &= \{(t, 0) \in \mathbb{R}^2 : 0 < t < 1\}, \\ \Gamma_2 &= \{(1, x) \in \mathbb{R}^2 : 0 < x < 1\}, \\ \Gamma_3 &= \{(t, t^\alpha) \in \mathbb{R}^2 : 0 < t < 1\}, \end{aligned}$$

(see Fig. 1).

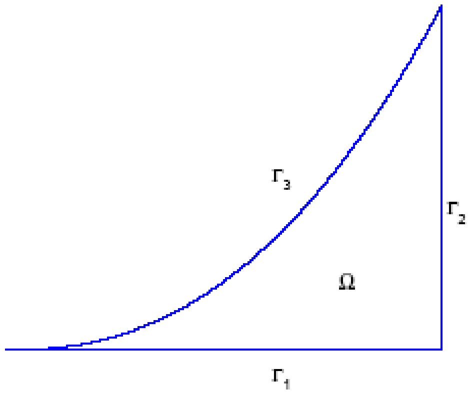


Fig.1: The time-varying domain Ω.

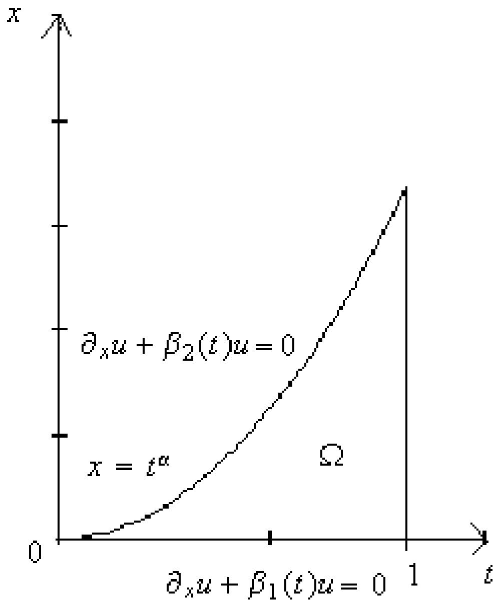
When $f = 0$ *and the variables* (t, x) *lie in a rectangular domain, Equation (1.2.1) represents Fick's second law which modelizes, for instance, the concentration (of atoms)* $u(t, x)$ *at time* t *in a position* x, *(like the carburization of steel) in a homogenous system (pure metal or any alloy). Besides being interesting in itself, Problem (1.2.1) governs, for example, the simplified diffusion equation of neutrons in the deviation situations from their trajectories. It is also the modelization of the lateral diffusion of a pollutant in a flow of river with variable width.*

Example 1.2.2 *Consider the same diffusion equation subject to Robin type boundary conditions instead of Dirichlet boundary conditions*

$$\begin{cases} \partial_t u - \partial_x^2 u = f \in L^p(\Omega), \\ \partial_x u + \beta_i(t)\, u_{/\Gamma_1 \cup \Gamma_2} = 0, \; i = 1, \, 2, \end{cases} \tag{1.2.2}$$

with

$$\Gamma_1 = \left\{ (t, 0) \in \mathbb{R}^2 : 0 < t < 1 \right\},$$
$$\Gamma_2 = \left\{ (t, t^\alpha) \in \mathbb{R}^2 : 0 < t < 1 \right\}, \; \alpha \geq 1/2.$$



$$\Omega = \{(t,x) \in \mathbb{R}^2 : 0 < t < 1, 0 < x < t^\alpha\}.$$

Such boundary conditions also occur in practical applications. Here, Problem (1.2.2) can also governs lateral diffusion of a pollutant in a flow of river with variable width and the boundary lateral conditions mean that the flux of diffusion $\partial_x u(t,x)$ *is proportionnal to the density of* $u(t,x)$ *at each time* t *.*

1.3 Some methods of resolution

In this section, we describe two powerful methods which have been successfully applied for solving parabolic problems in non-cylindrical domains, namely, the Rothe's method and the sum of operators method.

1.3.1 Rothe's method

One of the classical methods which has been successfully applied for solving parabolic problems in non-cylindrical domains is the so-called method of lines, where the derivatives with respect to one variable are replaced by difference quotients which finally leads to systems of differential equations for functions of the remaining variables. If we replace the time derivative $\partial_t u$ for the parabolic equation

$$\partial_t u(x,t) + A(x,t)\, u(x,t) = f(x,t), \tag{1.3.1}$$

in the cylindrical domain $Q = \Omega \times]0,T[$, where $\Omega \subset \mathbb{R}^N$ and A is an elliptic operator, then this method is called Rothe's method or time discretization. Using time discretization, evolution problem (1.3.1) is approximated by corresponding elliptic problem by means of which an approximate solution for the original evolution problem is constructed. Indeed, we devide the interval $]0,T[$ into n subintervals of the length $h = T/n$ and denote

$$z_k = z_k(x) = u(x, kh),\ x \in \Omega,\ k = 1, 2, ..., n.$$

Then we can consider, after replacing the derivative $\partial_t u$ for $t \in [(k-1)h, kh]$ by

$$(z_k(x) - z_{k-1}(x))/h,$$

the following system of n differential equations in x for the unknown functions $z_k(x)$, $k = 1, 2, ..., n$:

$$\frac{z_k(x) - z_{k-1}(x)}{h} - A(x, kh) = f(x, kh),\ x \in \Omega, \tag{1.3.2}$$

where we start with some given initial condition $z_0(x) = u_0(x)$ (and boundary conditions). The solutions $z_k(x)$ of (1.3.2) are solutions of (1.3.1) at the discrete values $t = kh$. Using these functions we can construct the Rothe function $u_n(x,t)$ which is defined in the interval $]0,T[$ by

$$u_n(x,t) = z_{k-1}(x) + \frac{t-(k-1)h}{h}(z_k(x) - z_{k-1}(x)),\ t \in [(k-1)h, kh],\ k = 1, 2, ..., n.$$

Note that $u_n(x,t)$ is a piecewise linear function of t, and it can be shown that $u_n(x,t)$ converges (in some appropriate sense) to the solution of (1.3.1) as $n \longrightarrow \infty$.

Rothe's method was introduced by the German mathematician E. Rothe in the year 1930 for solving second order linear parabolic equations with one space variable (see [55]). Since Rothe's classical paper [55] this method has been successfully applied to various problems: linear as well as nonlinear, parabolic and hyperbolic too. Linear equations of higher orders were considered by Ladyženskaja [34]. J. Kacur in [24], [25] has investigated non-linear evolution equations of parabolic type. For hyperbolic equations see e.g. [46] and [68].

We note that the method of Rothe in all works just mentioned was considered only for cylindrical domains. However, Rothe's classical method was extended so that it can be used to solve some linear parabolic boundary value problems in non-cylindrical domains. Let us mention, for example, J. L. Lions [44], who used a rather abstract approach and the papers [12], [29] where some fainly new results can be found. Hereafter we will give an example (see [29]) which illustrate this extension to non-cylindrical domains. Let us consider the following problem

$$\begin{cases} \partial_t u - \Delta u = f(x,t) \text{ in } Q \\ u = 0 \text{ on } \partial Q \setminus \Omega_T \end{cases} \tag{1.3.3}$$

in the non-cylindrical domain

$$Q = \{(x,t) : x = (x_1, x_2, ... x_N) \in \Omega_t,\ 0 < t < T\}$$

where $(0,T)$ is a finite interval, $\Omega_t \in C^{0,1}\left(\mathbb{R}^N\right)$ (here, $C^{0,1}\left(\mathbb{R}^N\right)$ is a set of all bounded domains in $\mathbb{R}^N$, whose boundary can be locally described by a function from $C^{0,1}(K)$, where $K \subset \mathbb{R}^{N-1}$ is a cube for every $t, s \in (0,T)$, $t < s$

$$\emptyset \neq \Omega_0 \subset \Omega_t \subset \Omega_s \subset \Omega_T$$

(see Fig. 3).

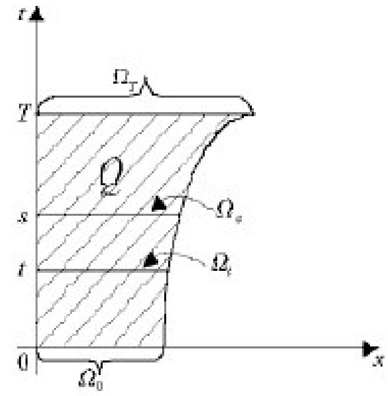


Fig. 3: The non-cylindrical domain Q.

By using an extension of the Rothe's method described above, the following result is proved in [29]

Theorem 1.3.1 *Assuming that there exists a function*

$$F \in V^1\left(0,T;L^2\left(\Omega_T\right)\right) \cap C\left(0,T;L^2\left(\Omega_T\right)\right)$$

such that

$$F\left(x,t\right) = f\left(x,t\right) \text{ for all } \left(x,t\right) \in Q$$

with

$$V^1\left(0,T;H\right) = \left\{ w\left(t\right) : \sup_D \sum_{k=1}^{J} \left\| w\left(t_k\right) - w\left(t_{k-1}\right)\right\|_H < \infty \right\}$$

where the supremum is taken over all decompositions $D = \{t_1, t_2, ..., t_J\}$ of $(0,T)$. Then Problem (1.3.3) has exactely one weak solution, i.e. a function which is a weak limit of Rothe's functions $u_n(x,t)$ in the space $L^2(0,T;V_T)$, where

$$V_T = \left\{ v \in H^{1,2}\left(\Omega_T\right) : v = 0 \text{ on } \partial\Omega_T \right\} = H_0^{1,2}\left(\Omega_T\right)$$

with

$$H^{1,2}\left(\Omega_T\right) = \left\{ u \in L^2\left(\Omega_T\right) : \partial_t u, \partial_{x_1}^i \partial_{x_2}^j ... \partial_{x_N}^k u \in L^2\left(\Omega_T\right), 1 \le i+j+k \le 2 \right\}.$$

1.3.2 Sum of operators method

In this subsection, we recall the main features of the $A+B$ method, and give an example which shows how this method can be applied to parabolic equations in non-cylindrical domains.

In 1975, P. Grisvard in collaboration with G. Da Prato [11] developed an abstract method (the sum of operators' method) to study the equation

$$Au + Bu = f \tag{1.3.4}$$

where $A : D(A) \subset X \longrightarrow X$ and $B : D(B) \subset X \longrightarrow X$ are two closed linear operators in a complex Banach space X and f is a given element of X. The solution of (1.3.4) was represented by a Dunford integral containing the resolvents $z \longmapsto (z-A)^{-1}$ and $z \longmapsto (z+B)^{-1}$ and its regularity was studied by using the interpolation spaces between $D(A)$ or $(D(B))$ and X: the main result of this method was the so called maximal regularity, i.e., when Au and Bu belong to the same space where f is prescribed.

Let us now recall the main features of this method: Assume that both operators satisfy the following assumptions of Da Prato-Grisvard type [11].

There exist positive numbers r, M_A, M_B, θ_A, θ_B such that

$$\theta_A + \theta_B < \pi, \tag{1.3.5}$$

$$\begin{aligned} &\rho(-A) \supset \Sigma_{\pi-\theta_A} := \{z \in \mathbb{C} : |z| \geq r, |\arg z| < \pi - \theta_A\} \text{ and} \\ &\forall\lambda \in \Sigma_{\pi-\theta_A}, \quad \left\|(A+\lambda I)^{-1}\right\|_{L(E)} \leq \frac{M_A}{|\lambda|}, \end{aligned} \tag{1.3.6}$$

$$\begin{aligned} &\rho(-B) \supset \Sigma_{\pi-\theta_B} := \{z \in \mathbb{C} : |z| \geq r, |\arg z| < \pi - \theta_B\} \text{ and} \\ &\forall\mu \in \Sigma_{\pi-\theta_B}, \quad \left\|(B+\mu I)^{-1}\right\|_{L(E)} \leq \frac{M_B}{|\lambda|}. \end{aligned} \tag{1.3.7}$$

We also assume that there are constants $C > 0$, $\lambda_0 > 0$, (with $\lambda_0 \in \rho(-A)$), τ and ρ such that

$$\begin{cases} (i) \ \left\|(A+\lambda_0 I)(A+\lambda I)^{-1}\left[(A+\lambda_0 I)^{-1}; (B+\mu I)^{-1}\right]\right\|_{L(E)} \\ \leq \dfrac{C}{|\lambda|^{1-\tau} . |\mu|^{1+\rho}} \quad \forall\lambda \in \rho(-A), \forall\mu \in \rho(-B), \\ (ii) \ 0 \leq \tau < \rho \leq 1. \end{cases} \tag{1.3.8}$$

For any $\sigma \in]0,1[$ and $1 \leq p \leq +\infty$, let us introduce the real Banach interpolation spaces $D_A(\sigma,p)$ between $D(A)$ and E (or $D_B(\sigma,p)$ between $D(B)$ and E) which are characterized by

$$D_A(\sigma,p) = \left\{\xi \in E : t \longmapsto \left\|t^{\sigma} A(A-tI)^{-1}\xi\right\|_E \in L_*^p\right\},$$

where L_*^p denotes the space of p-integrable functions on $(0,+\infty)$ with the Haar measure dt/t. For $p=+\infty$,

$$D_A(\sigma,+\infty) = \left\{\xi \in E : \sup_{t>0} \left\|t^{\sigma} A(A-tI)^{-1}\xi\right\|_E < \infty\right\}.$$

Then the main result proved in Labbas-Terreni [30] is

Theorem 1.3.2 *Under the assumptions (1.3.5), (1.3.6), (1.3.7) and (1.3.8), there exists λ^* such that for any $\lambda \geq \lambda^*$ and for any $h \in D_A(\sigma,p)$, equation $Aw+Bw+\lambda w = h$, has a unique solution $w \in D(A)\cap D(B)$ with the regularities Aw, $Bw \in D_A(\theta,p)$ and $Aw \in D_B(\theta,p)$ for any θ verifying $\theta \leq \min(\sigma,(\rho-\tau))$.*

A next step is to show how this method is used to solve the heat equation in domains with time-dependent boundaries. So, we revisite our problem given in Example (1.2.1) and we consider the model case $\varphi(t) = t^{\alpha}$, with the following hypothesis

$$\begin{cases} (i)\ 1/2 < \alpha < p-1, \\ (ii)\ p > \alpha/(2\alpha-1). \end{cases}$$

Let us introduce the following subspace of $L^p(\Omega)$

$$L^p_{t^{2\sigma\alpha}}\left(0,1;W^{2\sigma,p}_{t^{\alpha}}\right) = \left\{f \in L^p(\Omega) : \int_0^1 t^{2\sigma\alpha p}\int_0^{t^{\alpha}}\int_0^{t^{\alpha}} \frac{|f(t,x)-f(t,x')|^p}{|x-x'|^{2\sigma p+1}}dxdx'dt < \infty\right\},$$

then, the main result given in [32] is

Theorem 1.3.3 *For given $\sigma \in]0,1[$ such that $0 < \sigma < \dfrac{1}{2p}$ and $\sigma \leq \rho$, and for any $f \in L^p_{t^{2\sigma\alpha}}\left(0,1;W^{2\sigma,p}_{t^{\alpha}}\right)$, Problem (1.2.1) has a unique solution $u \in H_p^{1,2}(\Omega)$*

$$H_p^{1,2}(\Omega) = \left\{u \in L^p(\Omega) : \partial_t u, \partial_x^j u \in L^p(\Omega),\ j=1,2\right\}$$

with the regularities: u, $\partial_t u$, $\partial_x u$ and $\partial_x^2 u$ belong to $L^p_{t^{2\sigma\alpha}}\left(0,1;W^{2\sigma,p}_{t^{\alpha}}\right)$.

Proof. The principal idea to solve Problem (1.2.1) consists in transforming the parabolic equation in the non-cylindrical domain Ω into a variable-coefficient equation in a cylindrical domain.The change of variables

$$(t,x) \quad \mapsto \quad (t,y) = (t, x/t^{\alpha})$$

transforms Ω into the square $Q =]0,1[\times]0,1[$. Putting $u(t,x) = v(t,y)$ and $f(t,x) = g(t,y)$, then Problem (1.2.1) is transformed, in Q, into the degenerate evolution problem

$$\begin{cases} t^{2\alpha}\partial_t v(t,y) - \partial_y^2 v(t,y) - \alpha t^{2\alpha-1} y \partial_y v(t,y) = t^{2\alpha} g(t,y) = h(t,y) \\ u_{/\partial Q \smallsetminus \Gamma_2} = 0. \end{cases}$$

It is easy to see that $f \in L^p(\Omega)$ if and only if $t^{\alpha/p} g \in L^p(Q)$, i.e., if and only if the function $h = t^{2\alpha} g$ lies in the closed subspace of $L^p(Q)$ defined by

$$E = \left\{ h \in L^p(0,1; L^p(0,1)) : t^{-2\alpha+(\alpha/p)} h \in L^p(0,1; L^p(0,1)) \right\}.$$

This space is equipped with the norm $\|h\|_E = \left\| t^{-2\alpha+(\alpha/p)} h \right\|_{L^p(0,1;L^p(0,1))}$.

Now, it suffices to verify the hypothesis of the operator's sum method and conclude by using Theorem 1.3.2. For more details, see [32]. ■