Introduction

In this thesis, we shall be concerned by the existence, the uniqueness and the maximal regularity of solutions for some second-order linear parabolic problems in non-regular domains in the framework of Sobolev spaces built on the Lebesgue space of square integrable functions L^2. Our results will be given for the heat equation (the simplest of the second-order linear parabolic equations), but, they can be generalized to a large class of parabolic equations, namely, to higher-order and semilinear parabolic equations.

It is well known that there are two main approaches for the study of boundary value problems in non-regular domains. We can impose conditions on the non-regular domains to obtain regular solutions, or we work directly in the non-regular domains and we obtain singular solutions. The first approach will be illustrated in this work by the study of a parabolic equation with Cauchy-Dirichlet boundary conditions in a non-regular domain of $\mathbb{R}^3$, in Chapter 3, and by the resolution of a parabolic problem with Robin type boundary conditions in a non-rectangular domain, see Chapter 4. The second approach will be illustrated by the analysis of the heat equation in a domain of $\mathbb{R}^3$ with edge, see Chapter 5.

Let us briefly indicate the contents of every chapter of this thesis.

In Chapter 1, first we present a general survey on some boundary value problems for parabolic equations posed in non-cylindrical domains. We start in the first section by giving some history elements of boundary value problems for parabolic equations in non-cylindrical domains. Then we present some physical applications which explain the origin of such problems. We end the chapter by giving in the third section some classical methods for the resolution of such problems.

In Chapter 2, we collect some basic definitions and results that we need to develop further arguments in the following chapters. In Section 2.1, we define the basic functional spaces, in which we will work. First, we begin by introducing the so-called anisotropic Sobolev spaces $H^{r,s}$ built on the Lebesgue space of square integrable functions L^2. These function spaces are the naturel ones adopted in the study of parabolic equations and are different from those in the study of elliptic equations since the space variable x and time variable t play different roles in parabolic equations. Here, we focus our attention to two particular cases, namely, the case $(r,s) = (1,2)$ and the case $s = 2r$. Then, we give basic properties of these spaces. After that, we recall some concepts of interpolation theory that are needed for our purposes. We end this section by giving the definition of some particular Sobolev spaces. In the second section of this chapter, we give results on some parabolic model problems.

In Chapter 3, we study the existence and the regularity of the solution of the two dimensional second-order parabolic equation with Cauchy-Dirichlet boundary conditions

$$\begin{cases} \partial_t u - \partial_{x_1}^2 u - \partial_{x_2}^2 u = f \text{ in } L^2(Q) \\ \\ u = 0 \text{ on } \partial Q \setminus \Gamma_T, \end{cases} \tag{0.0.1}$$

set in a non-regular domain

$$Q = \left\{(t,x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\right\} \times]0,b[$$

of $\mathbb{R}^3$. The boundary of Q is defined by

$$\partial Q = (\Gamma \times]0,b[) \cup (\Omega \times \{0\}) \cup (\Omega \times \{b\}),$$

Γ is the boundary of

$$\Omega = \left\{(t,x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\right\}.$$

Γ_T is the part of the boundary of Q where $t = T$. T and b are two positive numbers, while φ_1 and φ_2 are continuous real-valued functions defined on $[0,T]$, Lipschitz continuous on $]0,T[$, and such that

$$\varphi_1(t) < \varphi_2(t)$$

for $t \in]0,T[$, and with the fundamental hypothesis $\varphi_1(0) = \varphi_2(0)$ and $\varphi_1(T) = \varphi_2(T)$. The difficulty related to this kind of problems comes from this singular situation for evolution problems, i.e., φ_1 is allowed to coincide with φ_2 for $t = 0$ and for $t = T$, which prevent the domain Q to be transformed into a regular domain without the appearance of some degenerate terms in the parabolic equation, see, for example Sadallah [58]. In order to overcome this difficulty, we impose sufficient conditions on the functions of parametrization $(\varphi_i)_{i=1,2}$, that is,

$$\varphi_i'(t)(\varphi_2(t) - \varphi_1(t)) \quad \longrightarrow \quad 0 \quad \text{as } t \longrightarrow 0, \quad i = 1, 2,$$

and

$$\varphi_i'(t)(\varphi_2(t) - \varphi_1(t)) \quad \longrightarrow \quad 0 \quad \text{as } t \longrightarrow T, \quad i = 1, 2,$$

so that Problem (0.0.1) admits a (unique) solution in the anisotropic Sobolev space

$$H_0^{1,2}(Q) := \left\{ u \in H^{1,2}(Q) : u_{/\partial Q \setminus \Gamma_T} = 0 \right\}$$

with

$$H^{1,2}(Q) = \left\{ u \in L^2(Q) : \partial_t u, \partial_{x_1}^j u, \partial_{x_2}^j u, \partial_{x_1}\partial_{x_2} u \in L^2(Q), j = 1, 2 \right\}.$$

This result complements similar results obtained by Sadallah [58] in the one-space variable case for a 2m-parabolic operator. Such kind of parabolic problems has been investigated by Baderko [8], but the author has considered domains which can not include our type of domains. Further references on the analysis of parabolic problems in non-cylindrical domains are: Savaré [64], Aref'ev and Bagirov [5], Hofmann and Lewis [22], Labbas, Medeghri and Sadallah [32], [33], and Alkhutov [3]. There are many other works concerning boundary-value problems in non-smooth domains (see, for example, Grisvard [20] and the references therein).

The main idea to solve Problem (0.0.1) consists in transforming the parabolic equation in the non-regular domain Q into a variable-coefficient equation in a regular domain. However, in order to perform this, one must assume that $\varphi_1(0) < \varphi_2(0)$ and $\varphi_1(T) < \varphi_2(T)$. So, in Section 3.2 of this chapter, we prove that Problem (0.0.1) admits a (unique) solution when Q could be transformed into a regular domain by means of a regular change of variable, i.e., we suppose that $\varphi_1(0) < \varphi_2(0)$ and $\varphi_1(T) < \varphi_2(T)$. In Section 3.3, we approximate Q by a sequence (Q_{α_n}) of such domains and we establish an uniform estimate of the type

$$\|u_n\|_{H^{1,2}(Q_{\alpha_n})} \leq K \|f\|_{L^2(Q_{\alpha_n})},$$

where u_n is the solution of Problem (0.0.1) in Q_{α_n} and K is a constant independent of n. Finally, in Section 3.4 we take limits in (Q_{α_n}) in order to reach the domain Q.

Chapter 4 is devoted to the analysis of the following one dimensional linear parabolic equation

$$\partial_t u - c^2 \partial_x^2 u = f, \tag{0.0.2}$$

subject to Robin type conditions

$$\alpha \partial_x u + \beta u = 0 \tag{0.0.3}$$

on the lateral boundary, which correspond to the case

$$(\alpha, \beta) \neq (0, 0).$$

The coefficients c, α and β satisfy suitable non-degeneracy assumptions and possibly depend on the time variable. The right-hand side term f of the equation (0.0.2) is taken in L^2. The problem is set in a domain of the form

$$\Omega = \left\{(t, x) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x < \varphi_2(t)\right\},$$

where T is a positive number, while φ_1 and φ_2 are smooth functions. One of the main issues of this work is that the domain can possibly be non-rectangular, for instance, the singular case where $\varphi_1(0) = \varphi_2(0)$ is allowed. Note that Robin type conditions (0.0.3) are perturbations by β of the Neumann type conditions and it is clear that Dirichlet and Neumann type boundary conditions correspond to two extreme cases, namely, $\beta = \infty$ and $\beta = 0$, respectively. We can find in [26], [31], [32], [61] and [62] solvability results of this kind of problems with Dirichlet boundary conditions. Hofmann and Lewis [22], have considered the classical heat equation (i.e.,the case where $c = 1$) with Neumann boundary condition in non-cylindrical domains under some conditions of Lipschitz's type on the geometrical behavior of the boundary which cannot include our domain. The authors showed that the optimal L^p regularity holds for $p = 2$ and the situation gets progressively worse as p approaches 1. We can find in Savaré [64], an abstract study for parabolic problems with mixed (Dirichlet-Neumann) lateral boundary conditions. The boundary assumptions dealt with by the author exclude our domain. In this work, we consider the case of Robin type boundary condition, namely, the case where $(\alpha, \beta) \neq (0, 0)$, and we will prove well-posedness results for the problem (0.0.2)-(0.0.3), under suitable non-degeneracy assumptions on the coefficients (c, α, β) and the functions of parametrization $(\varphi_i)_{i=1,2}$. More precisely, we will prove that Problem (0.0.2)-(0.0.3) has a solution with optimal regularity, that is a solution u belonging to the anisotropic Sobolev space

$$H_{\gamma}^{1,2}(\Omega) := \left\{u \in H^{1,2}(\Omega) : \alpha \partial_x u + \beta u_{/\Gamma} = 0\right\}$$

with

$$H^{1,2}(\Omega) = \left\{u \in L^2(\Omega) : \partial_t u, \partial_x u, \partial_x^2 u \in L^2(\Omega)\right\}$$

and Γ is the lateral boundary of Ω.

For this purpose, we begin in Section 4.2 of this chapter by deriving some preliminary results we need to develop further arguments. In Section 4.3, there are two main steps. First, we prove that Problem (0.0.2)-(0.0.3) admits a (unique) solution in the case of a regular domain. Secondly, we approximate Ω by a sequence $(\Omega_n)_n$ of such regular domains and we establish an uniform estimate similar to that proved in the previous chapter, under some assumptions on the coefficients (c, α, β) and the functions of parametrization $(\varphi_i)_{i=1,2}$ which will allow us to pass to the limit in the case where T is small enough. In Section 4.4, we show that the obtained local in time result can be extended to a global in time one.

In Chapter 5, there are two main parts. The first part of this chapter, see Section 5.2, is concerned with the extension of solvability results for a parabolic equation, set in a non-convex polygon obtained in [60], to the case of a polyhedral domain with edge on the boundary. In Chapter 3, see also [26], we have proved that under some conditions on the functions of the parametrization of a three-dimensional domain, the solution of the heat equation is "regular". The domains considered there include all the convex polyhedral domains (see, Sadallah [59]), but not all the polyhedral domains.

At the present time there exists a comprehensive theory of boundary value problems for parabolic equations and systems with a smooth boundary. One of the central results of this theory consists in the fact that if the coefficients of the equation and of the boundary operators, their right-hand sides, and the boundary of the domain are sufficiently smooth (the initial and boundary conditions must also satisfy the so-called compatibility conditions), then the solution itself of the problem is correspondingly smooth, see [23], [1], [43] and [35]. The lack of boundary smoothness in these problems leads to the occurrence of singularities of the solution in the neighbourhood of non-regular points of the boundary.

Let G be a non-convex bounded polyhedral domain of $\mathbb{R}^3$. In G, we consider the boundary value problem

$$\begin{cases} \partial_t u - \partial_x^2 u - \partial_y^2 u = f \\ \\ u_{/\partial_p G} = 0, \end{cases} \tag{0.0.4}$$

where $\partial_p G$ is the parabolic boundary of G and $f \in L^2(G)$.

The solvability of this kind of problems in the case of one-dimensional space variable has been investigated, for instance, in Aref'ev and Bagirov [4], [5] and in Sadallah [60] where results concerning the behavior of the solution of the heat equation in various singular domains of $\mathbb{R}^2$ were obtained. Solvability results for parabolic equations in domains with edges can be found in [13] and [54]. The solutions of boundary value problems, when posed and solved in non-smooth domains like polygons and polyhedra, have singular parts which are described in terms of special functions depending on the geometry of the domain and on the differential operators (see, for example, [20], [21] and the references therein).

The aim of the first approach is two-fold: Firstly, we exhibit singularities which appear in the solution u of Problem (0.0.4). Secondly, we study their smoothness. More precisely, we prove that there exist two functions v and w such that $u = v + w$ where v belongs to the anisotropic Sobolev space

$$H^{1,2}(G) = \left\{ v \in L^2(G) : \partial_t v, \partial_x^j v, \partial_y^j v, \partial_{xy}^2 v \in L^2(G), j = 1, 2 \right\}$$

whereas the singular part w is in the space $H^{r,2r}(G)$ with $r < 3/4$, defined as an interpolation space between $H^{1,2}(G)$ and $L^2(G)$, (the Sobolev spaces $H^{r,2r}(G)$ are those defined in Lions and Magenes [43]).

Our interest in the second part of this chapter, see Section 5.3, is the regularity of the solutions of the heat equation posed in a non-cylindrical domain - subject to Dirichlet conditions on the lateral boundary- in terms of the regularity of the inhomogeneous initial Cauchy data.

This chapter is organized as follows. In Section 5.2, we begin by preliminaries where we define the non-convex polyhedral domain and the basic functional spaces, in which we will work. Then, we describe the asymptotic behavior of the solution in the neighborhood of an edge in a model domain which is the union of two parallelepipeds. We will show that the solution may be written as a sum of a function which is the solution of a problem of type (0.0.4) and an infinite number of functions which are solutions of an homogeneous problem related to the problem (0.0.4). The main result concerning the optimal regularity of the singular part w is presented in Theorem 5.3.2, that is,

$$w \in H^{r,2r}(G) \Longleftrightarrow r < 3/4.$$

The proof is based on the Fourier transform as well as on some properties of interpolation theory and the fractional powers of operators. In Section 5.3, by using some interpolation results we prove the regularity of the weak solution of the heat equation set in a cylindrical domain in terms of the initial data. Finally, we use this case to prove new results of regularity of the weak solutions of the heat equation in a domain which is the union of two cylinders.