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Étude de certains problèmes aux limites par des méthodes variationnelles

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Notations

In what follows, we use the following notations:

$\hookrightarrow$	if X and Y are two normed spaces, we write $X \hookrightarrow Y$ to mean that X is included in Y and that canonical injection of X in Y is continuous.
$\hookrightarrow\hookrightarrow$	to mean that X is included in Y and that the canonical injection of X in Y is compact.
X'	topological dual.
$(\ ,\ )$	scalar product.
$\langle .,. \rangle$	duality pairing.
$\mathrm{Supp}(u)$	support of the function u.
$u_n \to u$	$\{u_n\}$ converges to u in norm.
$u_n \rightharpoonup u$	$\{u_n\}$ converges weakly to u.
$L^p(I)$	$= \{u : I \to \mathbb{R} : u$ is measurable and $\int_I \lvert u(x)\rvert^p dx < +\infty\}$ with $1 \leq p < \infty$.
$\lVert u\rVert_{L^p}$	$= \left[\int_I \lvert u(x)\rvert^p dx \right]^{1/p}$.
$L^\infty(I)$	$= \{u : I \to \mathbb{R} : u$ is measurable and $\lvert u(x)\rvert \leq M$ a.e. in I for some constant $M\}$,

its norm is denoted by $\lVert u\rVert_{L^\infty} = \inf\{M;\ \lvert u(x)\rvert \leq M$ a.e. in $I\}$.

$W^{1,p}(I)$	$= \{u \in L^p(I);\ \exists g \in L^p(I)$ *such that* $\int_I u\varphi' = -\int_I g\varphi,\ \forall \varphi \in C_c^1(I)\}$.
$\lVert u\rVert_{W^{1,p}}$	$= \lVert u\rVert_{L^p} + \lVert u'\rVert_{L^p}$.
$C(I)$	space of continuous functions on I.
$AC(I)$	space of absolutely continuous functions on I.
$C_0^\infty(I)$	space of indefinitely differentiable functions with compact supports on I.
a.e.	almost everywhere.
i.e.	meaning.

Introduction

Variational principles play an important part in mathematics and the physical sciences for three main reasons: they (i) unify many diverse fields, (ii) lead to new theoretical results, and (iii) provide powerful methods of calculation. Thus, the well-known Euler-Lagrange principle can be used to derive field equations of many kinds, extremum principles lead to new estimates for important physical quantities. Many problems, however, are usually first posed in the form of differential equations, or more generally as operator equations, and there is no guarantee that an equivalent variational problem exists (see [1], [2].

In this work, we study existence of solutions for an impulsive boundary value problem on the half-line by using variational methods and critical point theory and some theorems of existence of fixed points of operators that are defined on a Hilbert space using weak Ekeland variational principle .

This work was presented in six chapters:

The first one is an introduction, and contains a review of some functional analysis tools as operators on Banach spaces, Sobolev spaces, Sobolev embedding, differential calculus for functionals, minimization principle, critical point theory,....

In the second chapter, an impulsive boundary value problem on the half-line is considered and existence of solutions is proved using minimization principle and mountain pass theorem. In this chapter, we consider the impulsive boundary value problem, denoted by (P):

$$\begin{cases} -(p(t)u'(t))' & = & f(t,u(t)), & \text{a.e. } t \geq 0,\ t \neq t_j \\ u(0) = u(+\infty) & = & 0, & \\ \triangle(p(t_j)u'(t_j)) & = & h(t_j)I_j(u(t_j)), & j \in \{1,2,\ldots\}, \end{cases}$$

where $f : [0,+\infty)\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, the coefficient $p : [0,+\infty) \longrightarrow (0,+\infty)$ satisfies $\frac{1}{p} \in L^1[0,+\infty)$, and

$$M = \int_0^{+\infty} \left(\int_t^{+\infty} \frac{1}{p(s)} ds \right) dt < +\infty.$$

Here $t_0 = 0 < t_1 < t_2 < \ldots < t_j < \ldots < t_m \to +\infty$, as $m \to \infty$, are the impulse points, while the impulsive functions $I_j : \mathbb{R} \longrightarrow \mathbb{R}$ are assumed continuous. Finally

$$\triangle(p(t_j)u'(t_j)) = p(t_j^+)u'(t_j^+) - p(t_j^-)u'(t_j^-),$$

where $u'(t_j^+) = \lim\limits_{t\to t_j^+} u'(t)$ and $u'(t_j^-) = \lim\limits_{t\to t_j^-} u'(t)$.
The first result is:
suppose that the following conditions hold:
(H_1) There exist a constant $\mu \in [0,1)$ and positive functions $a_1, b_1 \in L^1[0,+\infty)$ such that

$$|f(t,x)| \leq a_1(t)|x|^\mu + b_1(t), \text{ for a.e. } t \in [0,+\infty) \text{ and all } x \in \mathbb{R}.$$

(I_0) There exist constants $k > 0$ and $\gamma \in [0,1)$ such that

$$|I_j(s)| \leq k|s|^\gamma, \ \forall\, s \in \mathbb{R}, \forall\, j \in \{1,2,\ldots\}.$$

Then problem (P) has at least one weak solution.
The second result is:
assume that (I_0) holds both with
(H_2) there exist positive functions $a_2, b_2 \in L^1(0,+\infty)$ with $|a_2|_{L^1} < \frac{1}{d^2}$ and

$$|f(t,x)| \leq a_2(t)|x| + b_2(t), \text{ for a.e. } t \in [0,+\infty) \text{ and all } x \in \mathbb{R}.$$

Then problem (P) has at least one weak solution.
The third result is:
suppose that the following conditions hold:
(H_3) There exist positive functions φ, g such that $\varphi \in L^1((0,+\infty),\mathbb{R})$ and $g \in C(\mathbb{R},\mathbb{R})$ with

$$|f(t,x)| \leq \varphi(t)g(x), \text{ for a.e. } t \in [0,+\infty) \text{ and all } x \in \mathbb{R}.$$

(H_4) $\lim_{x\to 0} \frac{f(t,x)}{x} = 0$, uniformly in $t \geq 0$.
(H_5) There exist positive functions $c_1, c_2 \in L^1((0,+\infty),[0,+\infty))$ and $\sigma > 2$ such that

$$(a)\, F(t,x) \geq c_1(t)|x|^\sigma - c_2(t), \quad \text{for a.e. } \ t \geq 0 \ \text{ and all } \ x \in \mathbb{R},$$

$$(b)\, \sigma F(t,x) \leq x f(t,x), \quad \text{for a.e. } \ t \geq 0 \ \text{ and all } \ x \in \mathbb{R} \setminus \{0\},$$

where

$$F(t,u) = \int_0^u f(t,s)ds.$$

(I_1) There exists $0 < \gamma \leq 2$ such that

$$\gamma \int_0^x I_j(s)ds \geq x I_j(x) > 0, \ \forall\, x \in \mathbb{R} \setminus \{0\}, \ \forall\, j \in \{1,2,\ldots\}.$$

Then problem (P) has at least one nontrivial weak solution.
In the third chapter, a fourth-order boundary value problem on the half-line is considered and existence of solutions is proved using a minimization principle and the mountain pass theorem.
In this chapter, we consider the boundary value problem, denoted by (Q):

$$\begin{cases} u^{(4)}(t) - u''(t) + u(t) &= f(t,u(t)), \quad t \in [0,+\infty), \\ u(0) = u(+\infty) &= 0, \\ u''(0) = u''(+\infty) &= 0, \end{cases}$$

where $f \in C([0,+\infty) \times \mathbb{R},\ \mathbb{R})$.
The first result is:
suppose that the following conditions hold:
(F_1) There exist two constants $1 < \alpha < \beta < 2$ and two functions $a,\ b$ with $\frac{a}{p^\alpha} \in L^1([0,+\infty),[0,+\infty))$, $\frac{b}{p^\beta} \in L^1([0,+\infty),[0,+\infty))$ such that

$$|F(t,u)| \leq a(t)|u|^\alpha, \quad \forall (t,u) \in [0,+\infty) \times \mathbb{R},\ |u| \leq 1$$

and

$$|F(t,u)| \leq b(t)|u|^\beta, \quad \forall (t,u) \in [0,+\infty) \times \mathbb{R},\ |u| > 1.$$

(F_2) There exist an open bounded set $I \subset [0,+\infty)$ and two constants $\eta > 0$ and $0 < \gamma < 2$ such that

$$F(t,u) \geq \eta |u|^\gamma, \quad \forall (t,u) \in I \times \mathbb{R},\ |u| \leq 1.$$

Then problem (Q) has at least one nontrivial weak solution.
The second result is:
suppose that the following conditions hold:
$(F3)$ There exist nonnegative functions φ, g such that $g \in C(\mathbb{R}, [0, +\infty))$ with

$$|f(t,x)| \leq \varphi(t)g(x), \quad \text{for } \ t \in [0,+\infty) \ \text{ and all } \ x \in \mathbb{R},$$

and for any constant $R > 0$ there exists a nonnegative function ψ_R with $\varphi\psi_R \in L^1(0,+\infty)$ and

$$\sup\Big\{g(\frac{y}{p(t)}) : y \in [-R,R]\Big\} \leq \psi_R(t) \ \text{ for a.e. } \ t \geq 0.$$

$(F4)$

$$\frac{1}{a(t)}F\Big(t, \frac{1}{p(t)}x\Big) = o(|x|^2) \ \text{ as } \ x \longrightarrow 0$$

uniformly in $t \in [0,+\infty)$ for some function $a \in L^1(0,+\infty) \cap C[0,+\infty)$.
$(F5)$ There exists a positive function c_1 and a nonnegative function c_2 with $c_1, c_2 \in L^1(0,\infty)$, and $\mu > 2$ such that

$$(a)\, F(t,x) \geq c_1(t)|x|^\mu - c_2(t), \quad \text{for } \ t \geq 0, \ \forall\, x \in \mathbb{R} \setminus \{0\},$$

$$(b)\, \mu F(t,x) \leq x f(t,x), \quad \text{for } \ t \geq 0, \ \forall\, x \in \mathbb{R}.$$

Then problem (Q) has at least one nontrivial weak solution.
In the fourth chapter, we give a new fixed point theorem on Hilbert spaces by using weak Ekeland variational principle for potential operators, we'll generalize the results proved in [3].
Let $T : \overline{U} \to H$ be a compact potential operator, where U is an open convex and bounded subset of a Hilbert space H with $0 \in U$. If there exists a constant $C > 0$ such that

$$\int_0^1 (T(su), u)\, ds \leq \frac{1}{2}\|u\|^2 - C\|u\| \ \text{ for all } \ u \in \partial U,$$

then T has a fixed point in $\overline{U}$.
As a result, we find:
(1) Let $T : \overline{U} \to H$ be a compact potential operator, where U is an open convex and bounded subset of a Hilbert space H with $0 \in U$. If there exists a constant $C > 0$ such that

$$\int_0^1 \|T(su)\|\, ds \leq \frac{1}{2}\|u\| - C \ \text{ for all } \ u \in \partial U,$$

then T has a fixed point in $\overline{U}$.
(2) Let $T : H \to H$ be a compact potential operator. If there exist a bounded linear operator B on H with $\|B\| < 1$ and $v^* \in H$ satisfying

$$(T(su), u) \leq (B(su), u) + (v^*, u) \ \forall s \in (0,1), \ \forall u \in \partial B(0,R) \text{ for some } R > \frac{2\|v^*\|}{1 - \|B\|},$$

then T has a fixed point in $\overline{B}(0,R)$.
In the fifth chapter, we give a new fixed point theorem on Hilbert spaces for potential α-positively homogeneous operators by using weak Ekeland variational principle.
(1) Let $T : \overline{U} \to H$ be a compact potential operator and α-positively homogeneous, where U is an open and bounded subset of a Hilbert space H with $0 \in U$. If there exists a constant $C > 0$ such that

$$(T(u), u) \leq \frac{\alpha + 1}{2}\|u\|^2 - C\|u\| \ \text{ for all } \ u \in \partial U,$$

then T has a fixed point in $\overline{U}$.
(2) Let $T : \overline{U} \longrightarrow H$ be a compact potential operator and α-positively homogeneous, where U is an open and bounded subset of a Hilbert space H with $0 \in U$. If there exists a constant $C > 0$ such that

$$\|T(u)\| \leq \frac{\alpha + 1}{2}\|u\| - C \ \text{ for all } \ u \in \partial U,$$

then T has a fixed point in $\overline{U}$.
In the last chapter, we present some results in critical point theory without the Palais-Smale condition by using weak Ekeland variational principle that is an important tool in critical point theory and nonlinear analysis.
This result is:
let E be a reflexive Banach space, Ω be a bounded and weakly closed set of E with $J \in C^1(E, \mathbb{R})$, and let J' be strongly continuous on Ω. Suppose also that J satisfies $\|J'(\varphi(u))\|_{E^*} < k\|J'(u)\|_{E^*}$ for all $u \in \Omega$, where $\varphi : \Omega \longrightarrow \Omega$ is a function such that $\varphi(u) \neq u$ for all $u \in \Omega$, and $0 < k < 1$ is a constant . Then J has at least one critical point in Ω.

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